

Chaos in cylindrical stadium billiards via a generic nonlinear mechanism

Thomas Gilbert^{1,*} and David P. Sanders^{2,†}

¹*Center for Nonlinear Phenomena and Complex Systems,
Université Libre de Bruxelles, C. P. 231, Campus Plaine, B-1050 Brussels, Belgium*

²*Departamento de Física, Facultad de Ciencias, Universidad Nacional
Autónoma de México, Ciudad Universitaria, 04510 México D.F., Mexico*

We describe conditions under which higher-dimensional billiard models in bounded, convex regions are fully chaotic, generalizing the Bunimovich stadium to dimensions above two. An example is a three-dimensional stadium bounded by a cylinder and several planes; the combination of these elements may give rise to defocusing, allowing large chaotic regions in phase space. By studying families of marginally-stable periodic orbits that populate the residual part of phase space, we identify conditions under which a nonlinear instability mechanism arises in their vicinity. For particular geometries, this mechanism rather induces stable nonlinear oscillations, including in the form of whispering-gallery modes.

PACS numbers: 05.45.Jn, 05.20.-y, 45.50.Jf

Billiard models, in which a point particle moves freely between elastic collisions with a fixed boundary, are a fertile source of ideas in physics [1] and mathematics [2] alike. They provide a basis for some of the fundamental concepts of statistical mechanics, and are at the same time open to mathematically rigorous analysis [3]. In particular, they are some of the best-motivated models which exhibit strong chaotic dynamics.

There are two distinct categories of chaotic billiards. The first is the class of *dispersing* billiards, the prototypical example of which is the hard-sphere gas: the dynamics of N hard spheres in a three-dimensional box with elastic collisions is equivalent to a point particle in a $3N$ -dimensional space moving uniformly outside a collection of spherical cylinders, with specular reflections at the boundary [4]. The mechanism giving rise to chaos in such billiards is that of dispersion, where nearby trajectories separate at each collision with a convex surface; this leads to an overall exponential divergence and to a complete spectrum of Lyapunov exponents. The system is then said to be fully chaotic or *hyperbolic*.

The second category is made up of *defocusing* billiards, the most well-known example of which is the Bunimovich stadium [5]. Here, chaos is due to a mechanism different from dispersion, namely that of defocusing: the boundary of the stadium curves inwards with respect to the particle, so that nearby trajectories initially focus after colliding with this boundary; however, the distance to the next collision is typically longer than the distance to the focal point, so that they eventually defocus even more. This again leads to an overall exponential expansion in phase space and hence complete chaoticity.

Defocusing billiards have attracted much attention in the physics community, particularly in connection with quantum chaos [6], acoustic experiments in chaotic cavities [7], optical microcavity laser experiments [8, 9], and quantum conductance experiments [10], to name but a few.

The extent to which the defocusing mechanism works in

dimensions beyond two has, however, long remained unclear [11, 12]. In spite of some recent progress in addressing this problem [13–15], the models studied are rather non-generic, consisting of higher-dimensional planar surfaces with spherical caps [13], or of a three-dimensional cube with two mutually perpendicular half-cylindrical caps [14]; moreover, only the latter model is convex. Another example of three- and four-dimensional convex billiards with flat and spherical components thought to be chaotic was investigated in [16], but it has so far not proved amenable to a systematic treatment.

In this Letter, we consider cylindrical stadium billiards, by which we mean higher-dimensional convex billiards based on cylindrically-shaped structures cut by planar elements. We describe the conditions under which they can give rise to chaos through a conjunction of linear and nonlinear instabilities, thus demonstrating that hyperbolicity in defocusing billiards is easier to obtain than was previously believed.

A generic dynamical feature of these billiards is that periodic orbits come in continuous parametric families, which are associated to the flat directions of the cylinders parallel to their axes. Each such direction thus gives rise to a pair of parabolic eigenvalues in the eigenvalue spectrum of the periodic orbits. By studying the dynamics along the directions of the corresponding eigenvectors at a nonlinear level, we describe a mechanism by which families of periodic orbits can or cannot be stabilized along these directions. Applying this method to three-dimensional cylindrical stadium billiards, we establish the existence of a large class of fully-chaotic convex billiards where this mechanism acts as a repulsive force. On the contrary, when it acts as a restoring force, we identify stable modes of nonlinear oscillations, taking the form of bouncing-ball orbits and whispering-gallery modes.

The construction of our class of models is simplicity itself: in dimension three, they are formed by cutting a cylinder with one or more flat planes to form a convex region,

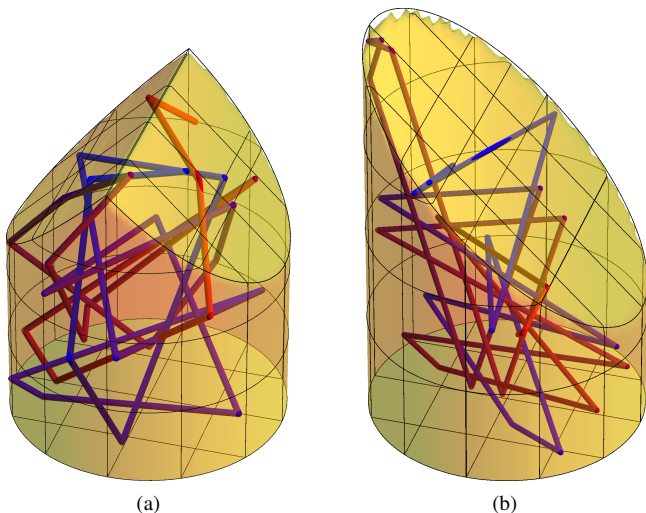


FIG. 1: (Color online) Three-dimensional stadia consisting of a circular cylinder cut by: (a) three planes, one perpendicular and two intersecting at right angles and cutting the cylinder at angle $\pi/4$; and (b) two planes, one perpendicular to the cylinder axis and the other at angle $\pi/4$. In each case, a typical trajectory is depicted, with varying shades as time progresses.

with the billiard dynamics taking place *inside* this region. Two simple examples, consisting of a three-dimensional circular cylinder cut by two or three planes, are shown in Fig. 1. In both cases, one of the planes is perpendicular to the cylinder axis, which just serves to confine the motion, without introducing any new dynamical features. The other planes, however, are angled away from perpendicular.

One might expect a cylindrical-shaped billiard to produce only integrable behavior. This would indeed be so if the planes were all perpendicular to the cylinder axis, leaving invariant the angular momentum along the axis. However, when one of the planes is oblique with respect to the cylinder axis, integrability is lost and defocusing can take place. It turns out that this is already sufficient to render the system of Fig. 1(a) completely chaotic throughout its range of parameter values. There remains, however, the possibility that stable oscillations take place in restricted phase-space regions, whereby the system loses ergodicity. Billiards like that of Fig. 1(b) may display such behavior: depending on the height h of the oblique plane above the cylinder base, nonlinearly stable oscillations are sometimes possible in confined regions of phase space. Examples of such oscillations are shown in Fig. 2, where we exploited the symmetry of the billiard to unfold it to a square cylindrical shape, by reflecting it multiple times in its planes,

In general, we call a *cylindrical stadium billiard* a bounded, convex region made by cutting a cylinder with flat planes, such that at least part of the boundary of the region is curved, and such that the symmetries of the system are broken. This construction easily extends to higher dimensions, as discussed below.

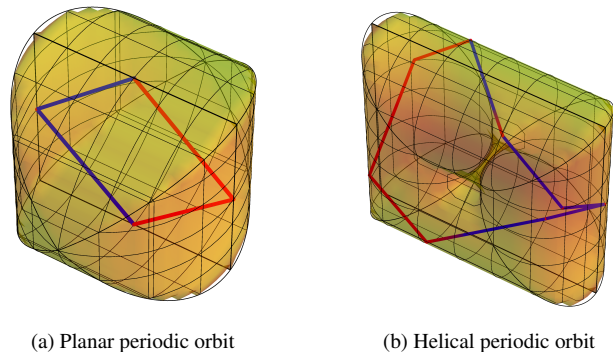


FIG. 2: (Color online) Nonlinearly stable oscillations occur for specific geometries of the three-dimensional stadium shown in Fig. 1(b). Two distinct types are shown: (a) Stable oscillations near the plane of symmetry of the billiard are found at negative height h for a large class of periodic orbits; (b) Approximations to helical periodic orbits whirl around the surface of the cylinder and can be stable in small regions of phase space.

In cylindrical billiards, the problem of determining the existence of stable elliptic islands is a fully nonlinear one. In the examples shown in Fig. 2, the phase space of the billiard map is four-dimensional and every periodic orbit has four eigenvalues, two of which are parabolic ($\equiv 1$), corresponding to motion along the orbit's family. The remaining pair can be either hyperbolic, in which case the orbit is unstable, or elliptic, in which case it is marginally stable. In order to assess the stability of the periodic orbit, it is then necessary to analyze the motion along the parabolic eigenvectors which, at a nonlinear level, is determined by the oscillations in the planes transverse to the cylinder axes, associated with the pairs of elliptic eigenvalues.

The two parabolic eigenvectors correspond to the displacement along a cylinder axis and its conjugate momentum. This displacement is identified as parameterizing a continuous family of periodic orbits; we denote the parameter by ε . Apart from exceptional cases—for example, the shape with negative cylinder height shown in Fig. 2(a)—there is typically a critical value of the parameter ε at which a bifurcation from elliptic to hyperbolic regimes is found. This occurs when the segments of the periodic orbit are sufficiently long that defocusing takes place.

In the absence of nonlinear effects, oscillations around a marginally-stable periodic orbit would always become unstable once the perturbed orbit crossed the bifurcation point. There is, however, a mechanism arising from the nonlinear corrections to the stability analysis, which can be generically identified by the properties of the eigenvalues of a quadratic form that drives the oscillations of the momentum along the cylinder axis, i.e., the rate of displacement along the family of periodic orbits. In the case of three-dimensional cylindrical billiards, letting w denote the momentum along the cylinder axis and θ and ξ the phase-space (Birkhoff) coordinates associated with the motion in

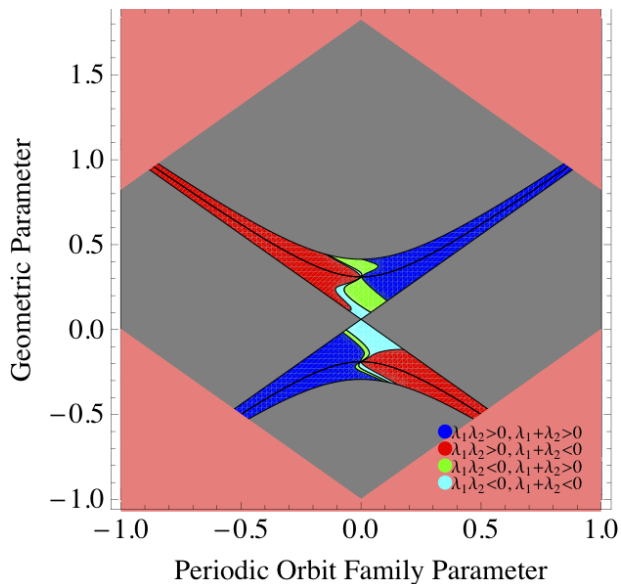


FIG. 3: (Color) Analysis of the eigenvalues of the quadratic form (2) allows to identify the precise regions in parameter space where the periodic orbit shown in Fig. 2(b) is stable. The vertical axis corresponds to the height h of the cylindrical billiard 1(b), measured at the lowest point of intersection of the oblique plane with the cylinder, and the horizontal axis to the parameter of the families of periodic orbits. Colors correspond to different regimes in parameter space: Light red indicates regions where the periodic orbit does not exist; gray codes hyperbolic regions, where the orbit is unstable; and the remaining colors characterize the quadratic form Q_w (2) in the region where the periodic orbits are marginally stable, according to the sign of its determinant (blue and red where positive; green and cyan where negative) and trace (blue and green where positive; red and cyan where negative). Similar results are found for the orbit shown in Fig. 2(a).

the transverse plane, the evolution along a period of the perturbation δw of the momentum takes the form

$$\delta w \mapsto \delta w + Q_w(\delta\theta, \delta\xi), \quad (1)$$

where $Q_w(\delta\theta, \delta\xi)$ is a quadratic form in the variables $\delta\theta$ and $\delta\xi$, which we write as

$$Q_w(\delta\theta, \delta\xi) = \begin{pmatrix} \delta\theta & \delta\xi \end{pmatrix} \begin{pmatrix} a_w & \frac{1}{2}b_w \\ \frac{1}{2}b_w & c_w \end{pmatrix} \begin{pmatrix} \delta\theta \\ \delta\xi \end{pmatrix}, \quad (2)$$

with coefficients a_w , b_w and c_w that are functions of the model's geometric parameter h and of the periodic orbit family parameter ε .

Depending on the nature of the periodic orbit, the displacement along the axis, which we interpret as a change in the parameter ε , may depend on θ and ξ through a linear combination, or through a quadratic form similar to (2). It is in any case a linear function of w , with a coefficient which is independent of ε . Given that oscillations take place due to the motion in the θ - ξ plane, nonlinear stability occurs whenever the perturbations along the coordinate w , Eq. (1), act as a restoring force to counteract these oscillations, so that they remain confined to the elliptic range of

the family. When these perturbations amplify the θ - ξ oscillations, on the other hand, the nonlinear mechanism acts as a repulsive force and destabilizes the family of periodic orbits.

These regimes can be analyzed in terms of the properties of the eigenvalues of the matrix in Eq. (2). Having fixed the geometry of the model, an interval of nonlinear stability in the family of periodic orbits corresponds to values of the determinant and trace of the quadratic form (2) such that w takes the sign opposite to that of the changes in the parameter ε . The results of this analysis are shown in Fig. 3 for the class of orbits shown in Fig. 2(b). The regions which are found to be elliptic from the linear stability analysis are divided into two symmetric tongues. The nonlinear analysis, however, shows that the upper tongue is unstable, and the lower one is stable only near the middle of the range of the family, $\varepsilon = 0$.

A detailed discussion may be found in [17]. There, it is shown that the periodic orbits under consideration are the crudest approximation to a family of helical periodic orbits, whose periods increase to infinity and which approximate an exact helix. The finite-approximation helical periodic orbits all have small regions of stability, whose sizes in parameter space decrease in direct proportion to the inverse of their periods. They can be interpreted as a sequence of whispering-gallery modes of the billiard, indexed by their periods. The nonlinear stability analysis of the planar orbits shown in Fig. 2(a) yields very similar results.

These stable oscillations can, however, easily be destroyed by changing the geometry of the billiard. In particular, the billiard with two oblique planes at right angle shown in Fig. 1(a) does not admit nonlinearly stable oscillations like those reported above, essentially because the perturbations will make collisions with both oblique planes, causing them to become unstable. It is thus an example of an ergodic, completely chaotic billiard in its whole parameter range. That is, the Lyapunov exponents, which measure the separation rate of nearby trajectories, of almost every point in the 6-dimensional phase space of Cartesian coordinates are constant and nonzero, except for two zero exponents associated with energy conservation and the corresponding time translation. This claim is substantiated by the results of numerical computations of these exponents, such as those shown in Fig. 4.

Examples of the general class of billiards described above arise naturally in models of interacting particles with flat surfaces, such as studied in [18] in the context of heat conduction. For example, the model of Fig. 1(a) is equivalent to a diatomic molecule with an interaction mediated by a massless string, and confined in an infinite two-dimensional channel. A systematic exposition of this equivalence is given in Ref. [17].

The generalization to higher-dimensional cylindrical billiards is straightforward. For example, the same molecule confined to a square box is equivalent to a cylinder in four dimensions with a two-dimensional circular base and two

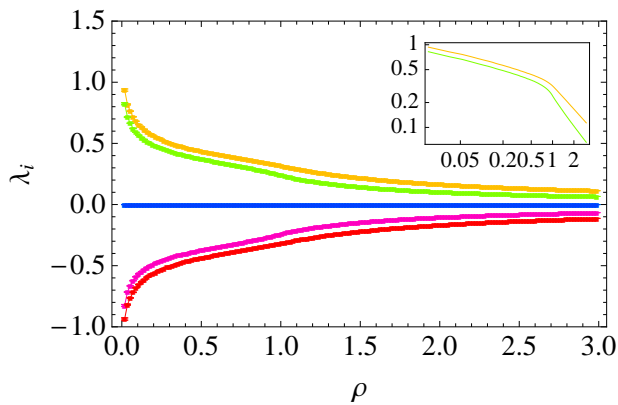


FIG. 4: (Color online) Spectrum of Lyapunov exponents of the billiard shown in Fig. 1(a). The parameter ρ measures the ratio between the radius of the cylinder and half its height. There are four non-zero exponents, arranged in two positive-negative pairs. The inset shows a log-log plot of the positive exponents. As ρ increases, the exponents decrease with the frequency of collisions with the oblique planes.

perpendicular axes, cut by eight hyperplanes. Similarly, a three-atom molecule corresponds to a billiard in four dimensions inside the Cartesian product of two disks [19]. In general, higher-dimensional cylindrical billiards cut by hyperplanes are expected to be chaotic. The existence of stable regions is rather exceptional, being found only for systems admitting families of periodic orbits with segments that remain close enough to the billiard surface that defocusing does not take place, and where the generic nonlinear mechanism identified above comes into play as a restoring force, rather than a repulsive one.

A main conclusion of our work is that cylindrical stadium billiards obtained by breaking the symmetry of cylindrical structures by the insertion of oblique planes easily yield fully chaotic dynamics, in spite of the existence of marginally-stable regions in phase space. A nonlinear mechanism operating along the the flat directions of the cylindrical surfaces may act as a repulsive force and prevent elliptic periodic orbits from giving rise to stable oscillations in their vicinity. In future work, we will explore the effect of this mechanism on cylindrical billiards with different forms of bases; preliminary results suggest that even cylindrical billiards with an elliptic base can be chaotic.

An interesting perspective is that our billiards can be used as building blocks for spatially-extended periodic and non-periodic structures, related to so-called track billiards [20]. These structures have straight segments and are angled with respect to one another. Our results allow to identify geometries such that classical dynamics within such structures is chaotic. This can be expected, for instance, in a series of straight tubes connected by joins. Such extended structures display diffusive behavior, as will be reported in future work.

The existence of such chaotic structures has further inter-

esting implications for physical systems and many potential applications, whether in nanostructures, fluid dynamics, acoustic devices, or optical fibers, where experiments would be possible.

The authors thank Carlangelo Liverani and Vered Rom-Kedar for helpful discussions. This research benefited from the joint support of FNRS (Belgium) and CONACYT (Mexico) through a bilateral collaboration project. The work of TG is financially supported by the Belgian Federal Government under the Inter-university Attraction Pole project NOSY P06/02. TG is financially supported by the Fonds de la Recherche Scientifique F.R.S.-FNRS. DPS acknowledges financial support from DGAPA-UNAM grant IN105209.

* Electronic address: thomas.gilbert@ulb.ac.be

† Electronic address: dps@ciencias.unam.mx

- [1] P. Gaspard, *Chaos, Scattering and Statistical Mechanics* (Cambridge University Press 1998).
- [2] Ya. G. Sinai, Proceedings of the International Congress of Mathematicians, Kyoto, Japan, August 21–29, 1990, ed. Ichiro Satake 242 (The Mathematical Society of Japan 1991).
- [3] D. Szász, *Hard Ball Systems and the Lorentz Gas* (Springer 2000).
- [4] N. Chernov and R. Markarian, *Chaotic Billiards* (American Mathematical Society, Providence, RI 2006).
- [5] L. A. Bunimovich, *Functional Analysis and Its Applications* **8**, 254 (1974); L. A. Bunimovich, *Commun. Math. Phys.* **65**, 295 (1979).
- [6] H.-J. Stöckmann, *Quantum Chaos: An Introduction* (Cambridge University Press 1999).
- [7] C. Draeger and M. Fink, *Phys. Rev. Lett.* **79**, 407 (1997).
- [8] N. Friedman, A. Kaplan, D. Carasso, and N. Davidson, *Phys. Rev. Lett.* **86**, 1518 (2001).
- [9] T. Harayama, P. Davis, and K. S. Ikeda, *Phys. Rev. Lett.* **90**, 063901 (2003).
- [10] C. M. Marcus, A. J. Rimberg, R. M. Westervelt, P. F. Hopkins, and A. C. Gossard, *Phys. Rev. Lett.* **69**, 506 (1992).
- [11] L. A. Bunimovich, *Physica D* **33**, 58 (1988).
- [12] M. P. Wojtkowski, *Commun. Math. Phys.* **129**, 319 (1990).
- [13] L. A. Bunimovich and J. Rehacek, *Commun. Math. Phys.* **197**, 277 (1998).
- [14] T. Papenbrock, *Phys. Rev. E* **61**, 4626 (2000).
- [15] L. A. Bunimovich and G. D. Magno, *Commun. Math. Phys.* **262**, 17 (2006); L. A. Bunimovich and G. D. Magno, *Erg. Theory Dyn. Sys.* **28**, 1377 (2008).
- [16] L. A. Bunimovich, G. Casati, and I. Guarneri, *Phys. Rev. Lett.* **77**, 2941 (1996).
- [17] T. Gilbert and D. P. Sanders arXiv:???
- [18] T. Gilbert and R. Lefevre, *Phys. Rev. Lett.* **101**, 200601 (2008).
- [19] T. Papenbrock, *Phys. Rev. E* **61**, 1337 (2000).
- [20] L. A. Bunimovich and G. D. Magno, *Commun. Math. Phys.* **288**, 699 (2009).