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Spin-1/2 particle on a cylinder with radial magnetic field

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Abstract

We study the motion of a quantum charged particle, constrained on the surface of a cylinder, in the presence of a radial magnetic field. When the spin of the particle is neglected, the system essentially reduces to an infinite family of simple harmonic oscillators, equally spaced along the axis of the cylinder. Interestingly enough, it can be used as a quantum Fourier transformer, with convenient visual output. When the spin-1/2 of the particle is taken into account, a non-conventional perturbative analysis results in a recursive closed form for the corrections to the energy and the wavefunction, for all eigenstates, to all orders in the magnetic moment of the particle. A simple two-state system is also presented, the time evolution of which involves an approximate precession of the spin perpendicularly to the magnetic field. A number of plots highlight the findings while several three-dimensional animations have been made available on the web.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

The quantum mechanical description of the motion of charged particles in a magnetic field is a classic application of the theory, having given rise to some of its most striking results. Among them, the seminal analysis by Dirac [5], of the motion in the field of a magnetic monopole, continues to inspire decades after its inception, and motivates the study of similar quantum systems that share the characteristic of providing insights into the fundamentals without too much distraction by analytical complexity. Such systems are invaluable pedagogically, as they furnish a manageable, yet captivating testing ground of the fundamentals of the theory.

The problem of the motion of a non-relativistic quantum particle in a plane, in the presence of a perpendicular homogeneous magnetic field is presented in several textbooks (see, e.g., [13], ch. XV, p 456)—nevertheless, it seems to be the only standard example of this type available. The main purpose of this paper is to draw attention to the fact that the analogous problem for the cylinder is also manageable, even when augmented to include a spin-1/2. In the latter case, we also show how the use of the creation and annihilation operator machinery greatly simplifies the perturbative analysis of the problem, in comparison with the standard textbook procedure. We emphasize that, although a cylindrically radial magnetic field inevitably brings to mind a monopole distribution as its source, it can actually be approximated in the laboratory by a particular current distribution (see section 2.1).

Despite the simplicity of the problem and it being an obvious variation on the monopole theme, we have not been able to find a treatment in the literature. It is then our hope that the use of the above simple system will help enhance the exposition of this fascinating part of the theory. It should also be of interest in practical applications, such as constrained quantum mechanics and carbon nanotube physics.

Consider a classical charged particle, constrained to move on the surface of an infinite cylinder, in the presence of a radial magnetic field,

$$\vec{B}(\rho = a, \phi, z) = B_0 \hat{\rho}, \tag{1}$$

where a is the radius of the cylinder and B_0 is the field strength on its surface. As mentioned above, such a radial field can be thought to be produced by a homogeneous linear magnetic charge density or by an infinite solenoid with a particular surface current distribution. The equations of motion for the particle imply that

$$m\dot{v}_z = -qB_0 v_\phi, \qquad m\dot{v}_\phi = qB_0 v_z, \tag{2}$$

where m, q are the mass and charge of the particle respectively and $\{\hat{\rho}, \hat{\phi}, \hat{z}\}$ is a right-handed orthonormal basis. The solutions to (2) are two simultaneous oscillations: the momentum p_z of the particle oscillates like, say, $\cos(\omega t)$ (with $\omega = qB_0/mc$) while its angular momentum along the z-axis oscillates like $\sin(\omega t)$. Thus, the particle's kinetic energy oscillates between a linear and a rotational form, becoming, for example, purely rotational at the turning points of the oscillation along z.

We study, in this paper, the quantum mechanical version of the above problem, adding, at a later stage, a spin-1/2 to the particle. The treatment of the spinless case, contained in section 2, is exact—the problem separates and reduces to an infinite collection of harmonic oscillators along z. We find, nevertheless, the resulting quantum system particularly rich and with surprising properties—it functions, for example, as a quantum Fourier transformer with convenient visual output (see section 2.3). The addition of spin is treated perturbatively in section 3, with a non-conventional method that greatly simplifies the calculations. We are able to give recursion relations for the corrections to the wavefunctions and the energy to all orders, for all unperturbed eigenstates, and apply the results to compute second-order corrections to the ground state. Several plots highlight the findings. We also make available on the web several three-dimensional colour animations of the time evolution of the wavefunction, with or without spin, and corresponding to various initial conditions (see http://www.nuclecu.unam.mx/~chrss). The appendix shows how the standard perturbation theory treatment of the problem reproduces, albeit laboriously, our first-order results.

2. The spinless case: reduction to the harmonic oscillator

2.1. The magnetic field

Before analysing the problem presented above, we would like to point out that, apart from the obvious (but experimentally unattainable) realization of the magnetic field (1) by a linear magnetic monopole distribution, there exists actually an experimental setup that can approximate this field in the laboratory. Indeed, consider a cylindrical solenoid in the interior of the cylinder where the particle lives, with its axis along $\hat{z}$, carrying a surface current distribution $\vec{J}$ given by

$$\vec{J} = -\frac{2B_0 a}{\mu_0 R^2} z\hat{\phi}, \tag{3}$$

where $R < a$ is the radius of the solenoid. The field produced by the above $\vec{J}$ is

$$\vec{B}(\vec{r}) = \begin{cases} \dfrac{B_0 a}{R^2}(\rho, 0, -2z) & \text{for} \quad \rho < R \\ B_0 a\left(\dfrac{1}{\rho}, 0, 0\right) & \text{for} \quad \rho > R, \end{cases} \tag{4}$$

coinciding with that in (1) at $\rho = a$. That the field outside of the solenoid is radial can be deduced from symmetry arguments and the principle of superposition alone—we invite the reader to construct such a proof, considering the above current distribution as the superposition of an infinite sequence of semi-infinite solenoids, each carrying an infinitesimal *uniform* current density and displaced infinitesimally w.r.t. the previous one along $\hat{z}$.

The corresponding vector potential $\vec{A}$, such that $\vec{B} = \nabla \times \vec{A}$, is

$$\vec{A}(\vec{r}) = \begin{cases} -\dfrac{B_0 a}{R^2}(0, \rho z, 0) & \text{for} \quad \rho < R \\ -B_0 a\left(0, \dfrac{z}{\rho}, 0\right) & \text{for} \quad \rho > R. \end{cases} \tag{5}$$

The second line of (5) gives the functional form of $\vec{A}$ in the vicinity of the surface of the cylinder and is therefore the one that should be used in the Hamiltonian. Note that the line integral of $\vec{A}$ around a constant-z circle is nonzero and equal to the flux of the magnetic field through the disc bounded by the circle, the latter coming entirely from the field in the interior of the solenoid. Had we worked with a linear monopole density, instead of a current distribution, we could still have used the above form of $\vec{A}$, everywhere except along the z-axis, but then it would not be clear how to account for the nonzero line integral mentioned above.

The above might be mathematically sound, but infinite solenoids are hard to come by in a laboratory. What is still left to show is that truncating the solenoid down to a (sufficiently long) finite size does not essentially alter the above results. More precisely, we need to consider the contribution to the field at, say, $z = 0$, of the semi-infinite parts of the solenoid with $|z| > L$ and show that it can be made arbitrarily small for L sufficiently large. This follows easily from the fact that the field produced at the origin by a circular ring of small width Δz, located at z, falls off like z^{-2}, for z large enough (being the field of a dipole with current that increases like z). Integrating from $z = \pm L$ to $\pm\infty$ we find that the above contributions fall off like L^{-1} and hence a sufficiently long solenoid can approximate arbitrarily well, near its centre, the field given in (4).

2.2. The spectrum

The Hamiltonian for a quantum spinless particle constrained to move on the surface of the cylinder is given by

$$\begin{aligned}\hat{H} &= \frac{1}{2m}(\vec{p} - q\vec{A})^2 \\ &= -\frac{\hbar^2}{2m}\left(\frac{1}{a^2}\partial_\phi^2 + \partial_z^2\right) + \frac{q^2 B_0^2}{2m} z^2 - \mathrm{i}\frac{\hbar q B_0}{ma} z \partial_\phi .\end{aligned} \tag{6}$$

The wavefunction $\Psi(\phi, z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\ell\phi} Z(z)$ is an eigenfunction of $\hat{H}$, with eigenvalue E, provided $Z(z)$ satisfies (primes denote differentiation w.r.t. z)

$$-\frac{\hbar^2}{2m} Z''(z) + \frac{1}{2} m\omega^2 (z + \ell b)^2 Z(z) = E Z(z), \tag{7}$$

where we have set

$$\omega = \frac{q B_0}{m}, \qquad b = \frac{\hbar}{q B_0 a}, \tag{8}$$

and, in what follows, we take $\hbar = m = \omega = 1$. This is the equation for a simple harmonic oscillator (SHO), centred at $z = -\ell b$. We conclude that, for each integer value of ℓ, one obtains a copy of the usual SHO spectrum, with the eigenfunctions centred at $z = -\ell b$, i.e., the eigenfunctions and eigenvalues of $\hat{H}$ are given by

$$\begin{aligned}\langle \phi, z \mid n, \ell \rangle &= N_n H_n(z + \ell b)\, \mathrm{e}^{-(z+\ell b)^2/2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\ell\phi}, \\ E_{\ell,n} \equiv E_n &= n + \frac{1}{2}, \qquad N_n \equiv (2^n n! \sqrt{\pi})^{-\frac{1}{2}},\end{aligned} \tag{9}$$

where $|n, \ell\rangle$ denote the corresponding eigenkets ($n = 0, 1, 2, \ldots$; $\ell \in \mathbb{Z}$) and $H_n(z)$ are the Hermite polynomials. Note that from the second of (8) it follows that

$$ab = z_0^2, \tag{10}$$

where $z_0 \equiv (\hbar/m\omega)^{1/2}$ is the ground state Gaussian width. In the above units then, $z_0 = 1$ and $b = 1/a$.

2.3. A quantum Fourier transformer

Suppose that the wavefunction of the particle, at $t = 0$, is given by[3]

$$\Psi_\ell(\phi, z, t = 0) \equiv \langle \phi, z \mid 0, 0, \ell \rangle = N_0\, \mathrm{e}^{-z^2/2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\ell\phi}. \tag{11}$$

If $\ell = 0$, we have one of the infinitely many ground states of the system and the time evolution is by a phase factor. Consider now the case $\ell \neq 0$. Then the z-part 'sees' a quadratic potential centred at $z = -\ell b$ but the initial wavefunction is a Gaussian centred at the origin. This is a coherent state and its time evolution is an oscillation around $z = -\ell b$, with the frequency $\omega = 1$ of the oscillator,

$$\Psi_\ell(\phi, z, t) = N_0\, \mathrm{e}^{-\mathrm{i}t/2}\, \mathrm{e}^{\mathrm{i}\frac{\ell^2 b^2}{4} \sin 2t}\, \mathrm{e}^{-\mathrm{i}(z+\ell b)\ell b \sin t}\, \mathrm{e}^{-(z+\ell b(1-\cos t))^2/2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}\ell\phi}. \tag{12}$$

[3] We use the notation $\langle \phi, z \mid n, \ell, m \rangle = N_n H_n(z + \ell b)\, \mathrm{e}^{-(z+\ell b)^2/2} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}m\phi}$ (not to be confused, hopefully, with the standard spherical symmetry notation)—these wavefunctions are eigenfunctions of $\hat{H}$ only when $m = \ell$, in which case they will be denoted by $|n, \ell\rangle$, as above.

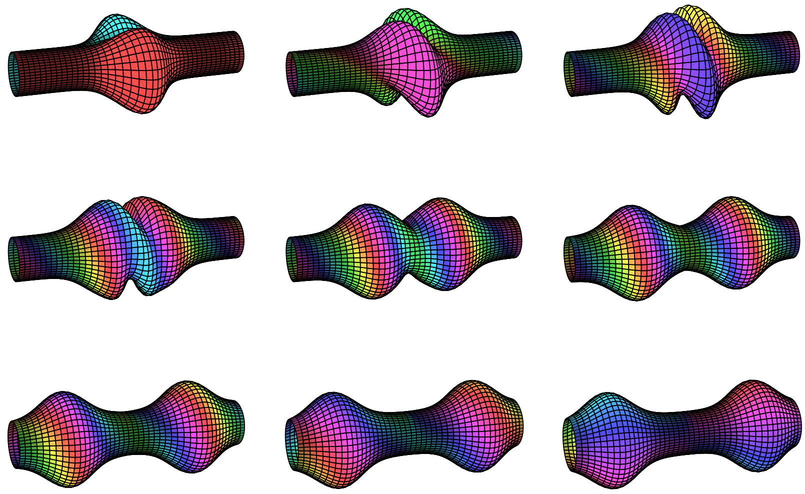

Figure 1. Time evolution of the initial wavefunction $\Psi(\phi, z, t = 0) = N_0\, \mathrm{e}^{-z^2/2} \cos\phi$. The modulus of Ψ is indicated by the radial distance of the surface from that of the cylinder while its phase is colour-coded, with $1, i, -1, -i$ corresponding to red, green, blue, purple (several animations in colour, including the above, can be seen at http://www.nuclecu.unam.mx/~ chryss). The time t is equal to zero at the top left and increases to the right and downwards, reaching $t = \pi$ (half a period) at the bottom right.

As long as one is interested in the physical characteristics of a single Ψ_ℓ, the first two phase factors, being z-independent, are irrelevant and are often omitted in the literature. In our case though, we will be dealing with superpositions of Ψ_ℓ for various ℓ, so the second (ℓ-dependent) phase factor will be important. We may now exploit linearity to write down the time evolution of a Gaussian (in z), centred at the origin, with arbitrary ϕ-dependence,

$$\Psi(\phi, z, t = 0) = N_0\, \mathrm{e}^{-z^2/2} f(\phi), \qquad \text{with} \quad f(\phi) = \sum_{\ell=-\infty}^{\infty} f_\ell\, \mathrm{e}^{\mathrm{i}\ell\phi}. \tag{13}$$

We obtain

$$\Psi(\phi, z, t) = \sum_{\ell=-\infty}^{\infty} f_\ell \Psi_\ell(\phi, z, t), \tag{14}$$

i.e., each Fourier mode of $f(\phi)$ gives rise to a Gaussian in z, oscillating like a coherent state around $z = -\ell b$ with frequency $\omega = 1$. Taking $b \gg 1$, so that the various Gaussians separate after half a period, converts the system to a quantum Fourier transformer with convenient visual output: looking at the wavefunction at time $t = \pi$ (a half-period), one sees the above Gaussians at the (second) turning point of their oscillation, at $z = -2\ell b$, with their amplitudes proportional to the Fourier amplitudes f_ℓ. In figure 1, we plot nine frames of the time evolution of Ψ, when $f(\phi) = \cos\phi$, with a time increment $\Delta t = \pi/8$—the last frame, at $t = \pi$, clearly displays the Fourier content of f. Figure 2 corresponds to the initial wavefunction $\Psi(\phi, z, t = 0) \sim \mathrm{e}^{-z^2/2}(1 + 2\mathrm{e}^{-\mathrm{i}\phi} + 3\mathrm{e}^{-\mathrm{i}2\phi} + 4\mathrm{e}^{-\mathrm{i}3\phi})$.

3. The spin-1/2 case

3.1. Separation of variables

For a spin-1/2 particle, the wavefunction has two components, $\Psi_+(\phi, z)$, $\Psi_-(\phi, z)$, which we arrange in a column vector. The spin interacts with the magnetic field via $\hat{H}_{\mathrm{int}} = -\lambda \vec{S} \cdot \vec{B}$,

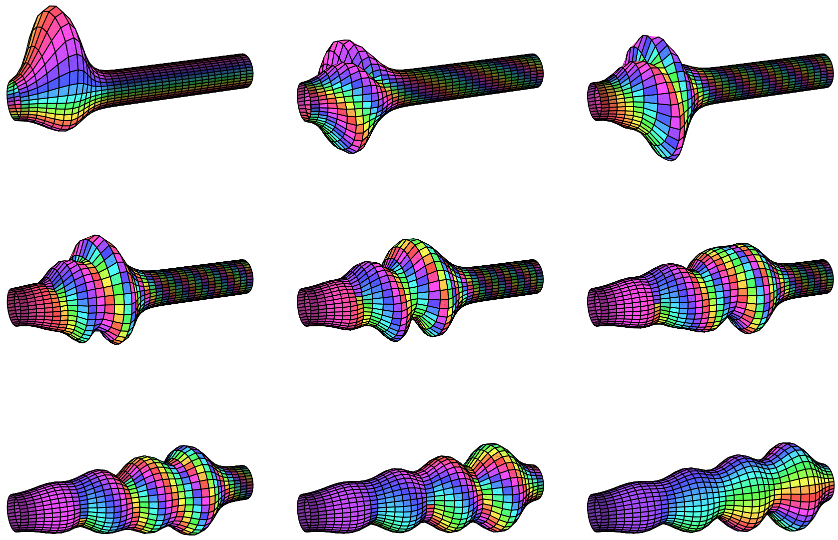

Figure 2. Time evolution of the initial wavefunction $\Psi(\phi, z, t=0) \sim e^{-z^2/2}(1 + 2e^{-i\phi} + 3e^{-i2\phi} + 4e^{-i3\phi})$. Conventions are as in figure 1. Note how the amplitudes of the Gaussians in the last frame correspond to the Fourier components of the initial wavefunction.

which, for the field given in (4) becomes

$$\hat{H}_{\rm int} = -\frac{1}{2}\lambda B_0 \begin{pmatrix} 0 & e^{-i\phi} \\ e^{i\phi} & 0 \end{pmatrix}, \qquad \lambda = \frac{gq}{2m}, \tag{15}$$

where g is the gyromagnetic factor of the particle. In order to achieve separation of variables now, we need to take

$$\Psi_+(\phi, z) = \frac{1}{\sqrt{2\pi}} e^{i\ell\phi} Z_+(z), \qquad \Psi_-(\phi, z) = \frac{1}{\sqrt{2\pi}} e^{i(\ell+1)\phi} Z_-(z). \tag{16}$$

The resulting equation for the Z is

$$\hat{H}_\ell Z_+(z) + \epsilon Z_-(z) = E Z_+(z) \qquad \hat{H}_{\ell+1} Z_-(z) + \epsilon Z_+(z) = E Z_-(z), \tag{17}$$

where $\hat{H}_\ell$ is a SHO Hamiltonian centred at $z = -\ell b$,

$$\hat{H}_\ell = -\tfrac{1}{2}\partial_z^2 + \tfrac{1}{2}(z + \ell b)^2, \tag{18}$$

and $\epsilon \equiv -\lambda B_0/2 = -g/4$, as follows from (8), (15). We see that the problem reduces to that of two SHOs, a distance b apart, coupled by the ϵ terms in (17). Our task is to solve (17) perturbatively in ϵ. Once the solutions are known, to a certain order in ϵ, we can form the spinor

$$\Psi(\phi, z) = \begin{pmatrix} \Psi_+(\phi, z) \\ \Psi_-(\phi, z) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2\pi}} e^{i\ell\phi} Z_+(z) \\ \frac{1}{\sqrt{2\pi}} e^{i(\ell+1)\phi} Z_+(z) \end{pmatrix}, \tag{19}$$

from which the probability density and spin direction can be extracted as

$$\rho = \sqrt{|Z_+|^2 + |Z_-|^2}, \qquad \alpha = 2\arctan\frac{|Z_-|}{|Z_+|}, \qquad \beta = \mathrm{Im}\left(\log\frac{\Psi_-}{\Psi_+}\right), \tag{20}$$

where the spin direction $\hat{n}$ is given by $\hat{n} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\alpha)$, in Cartesian coordinates. The solutions of (17) have no relative (complex) phase and can be taken as real.

Then, on the $\phi = 0$ plane, the spin lies in the x–z plane. The extra $e^{i\phi}$ factor in Ψ_- guarantees that when we change our position on the cylinder by a ϕ, the spin also rotates by the same angle and its direction is therefore obtained by a revolution of the $\phi = 0$ configuration, in other words, $\beta = \phi$ in (20). These remarks are of course valid only for the energy eigenstates—the time evolution of general states results in the spin pointing outside of the radial plane as well, even if they start within it.

3.2. Perturbative solution

One may treat the system of the two coupled differential equations in (17) by standard perturbation theory methods—we give the first-order analysis along these lines in the appendix. It is instructive though, as well as much more efficient, to exploit the SHO machinery of raising and lowering operators. We begin by transforming (17) into a single differential-difference equation. The solutions of (17) can be taken to satisfy $Z_-(z) = \pm Z_+(-z-b)$ (we will refer to the two possibilities as *symmetric* (Z_s) and *antisymmetric* (Z_a) respectively). This follows from the fact that the second of (17) is the mirror image, w.r.t. the point $z = -b/2$, of the first one. More formally, we may introduce the operator $\hat{M}$, which acts on functions of z according to $\hat{M}h(z) = h(-z-b)$. Note that $\hat{M}^2 = 1$ and that $\hat{M}H_0 = H_1\hat{M}$, $\hat{M}H_1 = H_0\hat{M}$. Taking $\ell = 0$, we write equations (17) in matrix form as

$$\mathbf{HZ} = E\mathbf{Z}, \qquad \mathbf{H} \equiv \begin{pmatrix} H_0 & \epsilon \\ \epsilon & H_1 \end{pmatrix}, \qquad \mathbf{Z} \equiv \begin{pmatrix} Z_+ \\ Z_- \end{pmatrix}. \tag{21}$$

Now observe that the Hamiltonian $\mathbf{H}$ commutes with $\mathbf{M} \equiv \left(\begin{smallmatrix} 0 & \hat{M} \\ \hat{M} & 0 \end{smallmatrix}\right)$. Since $\mathbf{M}^2 = \mathbf{1}$, its eigenvalues are ± 1, with the corresponding projectors being given by $\mathbf{P}_\pm = (\mathbf{1} \pm \mathbf{M})/2$ and satisfying $\mathbf{P}_\pm^2 = \mathbf{P}_\pm$, $\mathbf{P}_+\mathbf{P}_- = 0$, $\mathbf{P}_+ + \mathbf{P}_- = \mathbf{1}$. From (21) we then get $\mathbf{H}(\mathbf{P}_+ + \mathbf{P}_-)\mathbf{Z} = E\mathbf{Z}$. Multiplying from the left by $\mathbf{P}_+$ and using the fact that $\mathbf{H}$ commutes with $\mathbf{P}_\pm$, we find $\mathbf{H}(\mathbf{P}_+\mathbf{Z}) = E(\mathbf{P}_+\mathbf{Z})$ (similarly for $\mathbf{P}_-$). This shows that any solution $\mathbf{Z}$ of (21) can be written as the sum of two eigenfunctions of $\mathbf{M}$, *each of which* is also a solution of (21). Therefore, without loss of generality, we may assume $\mathbf{Z}$ to be an eigenvector of $\mathbf{M}$, leading to the above-mentioned symmetry between Z_+ and Z_-.

With symmetry considerations taken into account, we now turn to the solution of (17). Taking $\ell = 0$ and restricting to the symmetric case, the first of (17) becomes

$$\hat{H}_0 Z_{s+}(z) + \epsilon Z_{s+}(-z-b) = E Z_{s+}(z). \tag{22}$$

The form of this equation suggests treating the ϵ-term as a perturbation to the SHO Hamiltonian $\hat{H}_0$. Note however that this perturbation is not of the usual form of a potential function multiplying the wavefunction—rather, it involves the operator $\hat{M}$ introduced above (write the ϵ-term as $\epsilon\hat{M}Z_{s+}(z)$). Dealing, as we are, with a perturbative expansion, we would like to have ϵ small, the scale being set by the energy difference between neighbouring (unperturbed) eigenstates[4]. Since for SHO this energy difference is 1, for all eigenstates, we would like to have a system with $\epsilon \ll 1$. On the other hand, as has been already mentioned, substituting for λ in (15) the value $gq/(2m)$, with g depending on the particle, one finds $\epsilon = -g/4$. For an electron then, with $g \approx 2$, we get $\epsilon \approx -1/2$, which seems to imply that a perturbative analysis would be questionable in this case. One possibility is to leave it to our experimental colleagues to come up with a particle (system) with small enough g. As we will discuss at the end of section 3.3 though, this does not seem necessary, due to the fact that the matrix elements of $\hat{M}$ can be made small by increasing b. As a result, in our perturbative expansion

[4] We thank the (anonymous) editor for a clarifying remark on this point.

for the ground state in section 3.3, using $b = 2$, cubic corrections amount to less than 1% of the zeroth-order results, despite the fact that the value of ϵ used is $1/2$.

Write now $\hat{H}_0 = a^\dagger a + \frac{1}{2}$ and introduce the ket whose wavefunction is $Z_{s+}(z)$,

$$Z_{s+}(z) = \langle z|Z_{s+}\rangle, \qquad |Z_{s+}\rangle = \sum_{n=0}^{\infty} c_n (a^\dagger)^n |0\rangle_0 \equiv f(a^\dagger)|0\rangle_0, \tag{23}$$

where $|0\rangle_0$ is the ground state for ℓ equal to zero (i.e., centred at the origin) and $f(a^\dagger)$ is defined by the last equation. The idea is that any ket can be obtained by some function of $a^\dagger$ applied to the ground state—this follows from the fact that the set $\{|n\rangle\}$ forms a basis and $|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle$. The wavefunction $Z_{s+}(-z-b)$ is obtained from $Z_{s+}(z)$ by first reflecting around the origin and then effecting the translation $z \mapsto z + b$. Reflecting around the origin an eigenfunction of $\hat{H}_0$ introduces a sign given by the parity of the state, which shows that the reflected state is produced by $f(-a^\dagger)$ applied to the ground state. Using the identity

$$\mathrm{e}^{c(a-a^\dagger)} = \mathrm{e}^{-c^2}\, \mathrm{e}^{-ca^\dagger}\, \mathrm{e}^{ca}, \tag{24}$$

valid for any constant c, and the fact that the above translation in z is effected by the operator $\mathrm{e}^{\mathrm{i}b\hat{p}_z}$, we find for the reflected *and* translated state

$$\begin{aligned}\mathrm{e}^{\mathrm{i}b\hat{p}_z} f(-a^\dagger)|0\rangle_0 &= \mathrm{e}^{-b^2/4}\, \mathrm{e}^{-\frac{b}{\sqrt{2}}a^\dagger}\, \mathrm{e}^{\frac{b}{\sqrt{2}}a} f(-a^\dagger)|0\rangle_0 \\ &= \mathrm{e}^{-b^2/4}\, \mathrm{e}^{-\frac{b}{\sqrt{2}}a^\dagger} f\left(-a^\dagger - \frac{b}{\sqrt{2}}\right)|0\rangle_0.\end{aligned} \tag{25}$$

The last step above follows from the fact that the $a - a^\dagger$ commutation relations are identical to the $\partial_x - x$ ones, so that the operator $\mathrm{e}^{\frac{b}{\sqrt{2}}a}$, applied to functions of $a^\dagger$, effects the translation $a^\dagger \mapsto a^\dagger + b/\sqrt{2}$. Then (22) is brought into the form

$$\left(a^\dagger a + \frac{1}{2}\right) f(a^\dagger)|0\rangle_0 + \epsilon\, \mathrm{e}^{-b^2/4}\mathrm{e}^{-\frac{b}{\sqrt{2}}a^\dagger} f\left(-a^\dagger - \frac{b}{\sqrt{2}}\right)|0\rangle_0 = Ef(a^\dagger)|0\rangle_0. \tag{26}$$

From the above algebra isomorphism and the fact that $\partial_x f(x) = f'(x) + f(x)\partial_x$, we may also infer that $af(a^\dagger) = f'(a^\dagger) + f(a^\dagger)a$. Using additionally the fact that $a|n\rangle = 0$, (26) implies a differential-difference equation for the function $f(x)$,

$$xf'(x) + \frac{1}{2}f(x) + \epsilon\, \mathrm{e}^{-b^2/4}\mathrm{e}^{-\frac{b}{\sqrt{2}}x} f\left(-x - \frac{b}{\sqrt{2}}\right) = Ef(x). \tag{27}$$

We now specify to the case where the unperturbed state is the nth excited state of the SHO. The perturbed state will be denoted by $f_n(a^\dagger)|0\rangle_0$, with energy E_n, where

$$f_n(x) = \sum_{k=0}^{\infty} f_n^{(k)}(x)\epsilon^k, \qquad E_n = \sum_{k=0}^{\infty} E_n^{(k)}\epsilon^k. \tag{28}$$

Note that

$$f_n^{(0)}(x) = \frac{1}{\sqrt{n!}}x^n, \qquad E_n^{(0)} = n + \frac{1}{2}. \tag{29}$$

Substituting these expansions into (27) we obtain

$$\begin{aligned}x\partial_x f_n^{(k)}(x) - nf_n^{(k)}(x) &= -\mathrm{e}^{-b^2/4}\, \mathrm{e}^{-\frac{b}{\sqrt{2}}x} f_n^{(k-1)}\left(-x - \frac{b}{\sqrt{2}}\right) \\ &\quad + \frac{1}{\sqrt{n!}}E_n^{(k)}x^n + \sum_{m=1}^{k-1} E_n^{(k-m)} f_n^{(m)}(x),\end{aligned} \tag{30}$$

where we separated the $m = 0, k$ terms in the sum on the r.h.s and used (29). Note that the r.h.s. above only contains $f_n^{(m)}$ with $m < k$, so (30) can be used recursively to determine any $f_n^{(k)}(x)$.

The requirement that the perturbed eigenket be normalized implies that the corrections, order by order in ϵ, have to be orthogonal to the unperturbed eigenket $|n\rangle_0$. This in turn implies that, for $k > 0$, the coefficient of x^n in $f_n^{(k)}(x)$ must vanish. Then so does the coefficient of x^n in $x\partial_x f_n^{(k)}(x)$. Finally, the coefficient of x^n in the first term on the r.h.s. above can be computed by expanding both the exponential and $f_n^{(k-1)}\big(-x-\frac{b}{\sqrt{2}}\big)$ in a Taylor series around $x = 0$ and then multiplying the two series. Using the above, we can extract the coefficient of x^n on both sides of (30)—the resulting equation fixes recursively the energy corrections $E_n^{(k)}$,

$$E_n^{(k)} = \frac{1}{\sqrt{n!}}\,\mathrm{e}^{-b^2/4}\sum_{r=0}^{n}\binom{n}{r}\left(-\frac{b}{\sqrt{2}}\right)^{n-r}\left(\partial_x^r f_n^{(k-1)}\right)\left(-\frac{b}{\sqrt{2}}\right). \tag{31}$$

This is an appropriate point to comment on the antisymmetric solutions. The difference in this case is that the ϵ term in (27) appears with a minus sign. Since this is the only place where ϵ appears explicitly, we conclude that one gets the antisymmetric solutions from the symmetric ones by the substitution $\epsilon \to -\epsilon$. As we will see later on, symmetric solutions have their spin parallel, more or less, with the magnetic field while antisymmetric ones have it antiparallel. Since ϵ is proportional to the magnetic moment of the particle, the above statement about the relation between the two kinds of solutions essentially says that the symmetric solution for a particle coincides with the antisymmetric solution for the same particle but with the opposite magnetic moment.

3.3. Corrections to the ground state

For $n = 0$, equations (30), (31) simplify considerably,

$$x\partial_x f_0^{(k)}(x) = -\mathrm{e}^{-b^2/4}\,\mathrm{e}^{-\frac{b}{\sqrt{2}}x}\, f_0^{(k-1)}\left(-x-\frac{b}{\sqrt{2}}\right) + E_0^{(k)} + \sum_{m=1}^{k-1} E_0^{(k-m)} f_0^{(m)}(x) \tag{32}$$

$$E_0^{(k)} = \mathrm{e}^{-b^2/4} f_0^{(k-1)}\left(-\frac{b}{\sqrt{2}}\right). \tag{33}$$

The unitarity argument given above implies in this case that $f_0^{(k)}(0) = 0$, for all k greater than zero—this fixes the lower integration limit in the solution of (32) equal to zero. The change of variable $x_k' \to x s_k$ and the substitution of (33) finally give

$$\begin{aligned} f_0^{(k)}(x) = \mathrm{e}^{-b^2/4}\int_0^1 \frac{\mathrm{d}s_k}{s_k}\Bigg\{ & f_0^{(k-1)}\left(-\frac{b}{\sqrt{2}}\right) - \mathrm{e}^{-\frac{b}{\sqrt{2}}x s_k} f_0^{(k-1)}\left(-x s_k - \frac{b}{\sqrt{2}}\right) \\ & + \sum_{m=1}^{k-1} f_0^{(k-m-1)}\left(-\frac{b}{\sqrt{2}}\right) f_0^{(m)}(x s_k)\Bigg\}. \end{aligned} \tag{34}$$

Note that the (apparent) pole of the integrand at $s_k = 0$ cancels out.

We look in some detail now at the wavefunctions and resulting spin configurations, including up to quadratic corrections. For the first three $f^{(k)}$, equations (29), (34) give

$$f_0^{(0)}(x) = 1 \tag{35}$$

$$f_0^{(1)}(x) = \mathrm{e}^{-b^2/4}\int_0^1 \frac{\mathrm{d}s_1}{s_1}\left(1 - \mathrm{e}^{-\frac{b}{\sqrt{2}}x s_1}\right) \tag{36}$$

$$f_0^{(2)}(x) = \mathrm{e}^{-b^2/2}\int_0^1\int_0^1 \frac{\mathrm{d}s_2\,\mathrm{d}s_1}{s_2 s_1}\big\{2 - \mathrm{e}^{b^2 s_1/2} - \mathrm{e}^{-\frac{b}{\sqrt{2}}x s_2} + \mathrm{e}^{\frac{b^2}{2}s_1}\,\mathrm{e}^{-\frac{b}{\sqrt{2}}s_2(1-s_1)x} - \mathrm{e}^{-\frac{b}{\sqrt{2}}s_2 s_1 x}\big\}, \quad (37)$$

while for the corresponding energy corrections we get

$$E_0^{(0)} = \frac{1}{2}, \qquad E_0^{(1)} = \mathrm{e}^{-b^2/4}, \qquad E_0^{(2)} = \mathrm{e}^{-b^2/2}\int_0^1 \frac{\mathrm{d}s_1}{s_1}\big(1 - \mathrm{e}^{\frac{b^2}{2}s_1}\big). \quad (38)$$

In order to find the corrections to the wavefunctions, we need to apply the above f (with x replaced by $a^\dagger$) to the ground state and project the resulting ket onto the position eigenket $|z\rangle$, $Z_+^{(k)}(z) = \langle z|\, f_0^{(k)}(a^\dagger)|0\rangle$. For terms such as the second one in the integrand in (36), we use the identity (24) and the fact that $a|0\rangle = 0$—the results are

$$Z_+^{(0)}(z) = N_0\,\mathrm{e}^{-z^2/2} \quad (39)$$

$$Z_+^{(1)}(z) = N_0\,\mathrm{e}^{-b^2/4}\int_0^1 \frac{\mathrm{d}s_1}{s_1}\big(\mathrm{e}^{-z^2/2} - \mathrm{e}^{b^2 s_1^2/4}\,\mathrm{e}^{-(z+bs_1)^2/2}\big) \quad (40)$$

$$Z_+^{(2)}(z) = N_0\,\mathrm{e}^{-b^2/2}\int_0^1\int_0^1 \frac{\mathrm{d}s_2\,\mathrm{d}s_1}{s_2 s_1}\big\{\big(2 - \mathrm{e}^{b^2/2s_1}\big)\,\mathrm{e}^{-z^2/2} - \mathrm{e}^{b^2 s_2^2/4}\,\mathrm{e}^{-(z+bs_2)^2/2}$$
$$+\,\mathrm{e}^{b^2 s_1/2 + b^2 s_2^2(1-s_1)^2/4}\,\mathrm{e}^{-(z+bs_2(1-s_1))^2/2} - \mathrm{e}^{b^2 s_2^2 s_1^2/4}\,\mathrm{e}^{-(z+bs_2 s_1)^2/2}\big\}. \quad (41)$$

In terms of these, the symmetric solution for the spinor has components (we omit an overall normalization factor)

$$Z_{\mathrm{s}+}(z) = Z_+^{(0)}(z) + \epsilon Z_+^{(1)}(z) + \epsilon^2 Z_+^{(2)}(z), \qquad Z_{\mathrm{s}-}(z) = Z_{\mathrm{s}+}(-z-b) \quad (42)$$

and energy

$$E_{\mathrm{s}0} = E_0^{(0)} + \epsilon E_0^{(1)} + \epsilon^2 E_0^{(2)}, \quad (43)$$

while the antisymmetric solution is given by

$$Z_{\mathrm{a}+}(z) = -Z_+^{(0)}(z) + \epsilon Z_+^{(1)}(z) - \epsilon^2 Z_+^{(2)}(z), \qquad Z_{\mathrm{a}-}(z) = -Z_{\mathrm{a}+}(-z-b) \quad (44)$$

with energy

$$E_{\mathrm{a}0} = E_0^{(0)} - \epsilon E_0^{(1)} + \epsilon^2 E_0^{(2)}. \quad (45)$$

Plots of $Z_\pm$, for both cases, as well as the corresponding spin configurations, are given in figures 3 and 4. Note that the effect of the perturbation is small, despite the rather large value of $\epsilon = 0.5$. Roughly speaking, this can be traced to the fact that the value $b = 2$ used gives rise to a small overlap of two neighbouring ground state Gaussians. To clarify this point, we need to recall that, in standard perturbation theory, kth-order corrections are given as sums of products of k matrix elements of the perturbation in the unperturbed eigenkets. In our case, these matrix elements are

$$M_{mn} = \int_{-\infty}^{\infty} \mathrm{d}z\,\psi_m(z)\psi_n(-z-b), \quad (46)$$

where $\psi_m(z) = \langle z|m\rangle$. Using $f(a^\dagger) = (a^\dagger)^n/\sqrt{n!}$ in (25), we find

$$M_{mn} = \frac{(-1)^n}{\sqrt{n!}}\,\mathrm{e}^{-b^2/4}\,\langle m|\,\mathrm{e}^{-\frac{b}{\sqrt{2}}a^\dagger}\left(a^\dagger + \frac{b}{\sqrt{2}}\right)^n|0\rangle. \quad (47)$$

Expanding the exponential and the binomial and inspecting the terms that contribute to the matrix element, we conclude that M_{mn} is of the form $M_{mn} \sim \mathrm{e}^{-b^2/4}Q_{m+n}(b)$, where $Q_r(b)$

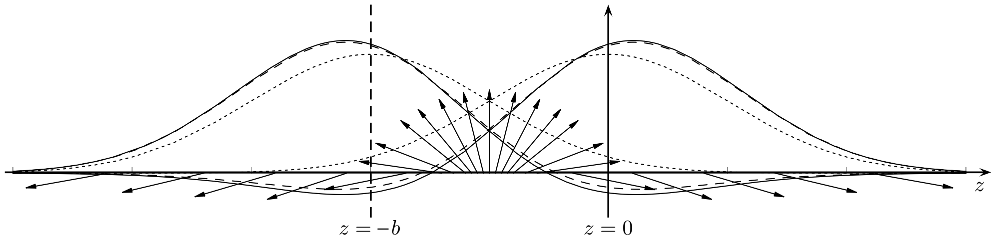

Figure 3. The spin components Z_{s+}, Z_{s-} and the corresponding spin configuration. The dotted curves give the zeroth-order result, i.e., two gaussians, centred at $z = 0$ and $z = -b = -2$. The dashed curves include corrections up to the first order while the solid ones up to the second ($\epsilon = 0.5$—a rather large value was used to make the effect visible). The integrals in equations (40), (41) have been evaluated numerically. The spin configuration shown includes quadratic corrections. Due to the symmetry, the spin points always along $\hat{\rho}$ (upwards in the figure) at $z = -b/2 = -1$, while it tends to $\pm\hat{z}$ as z tends to $\pm\infty$. The perturbation causes a zero in each component (for finite z)—the spin crosses the z-axis at those points. As a result, the two gaussians are pushed apart while the width of the central region, where the spin points up, is reduced.

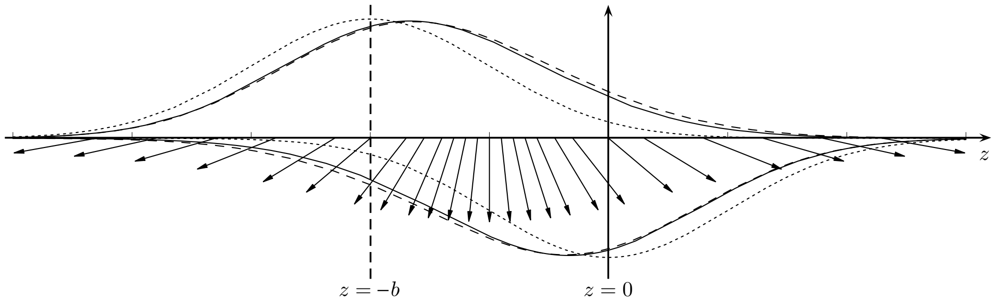

Figure 4. The spin components Z_{a+}, Z_{a-} and the corresponding spin configuration. Notation and parameter values are as in figure 3. Due to the antisymmetry, the spin points always along $-\hat{\rho}$ (downwards in the figure) at $z = -b/2 = -1$, while it tends to $\pm\hat{z}$ as z tends to $\pm\infty$. The perturbation pushes the two Gaussians together while the width of the region where the spin points downwards is increased.

denotes a polynomial of order r in b. It is clear then that $\lim_{b\to\infty} M_{mn} = 0$, so that, in the above limit, corrections of all orders tend to zero (for those values of ϵ for which the corrections are finite, for finite b). The precise way in which this happens, namely, the asymptotic expansion of the corrections for large b requires a detailed analysis which is beyond the scope of this paper. Since each matrix element is proportional to $e^{-b^2/4}$, one might suppose that the kth order corrections are proportional to the kth power of this exponential, which would suggest that the effective 'small parameter' of the perturbative expansion is $\epsilon e^{-b^2/4}$. This, however, is not the case: the infinite sums involved in the corrections, once the above exponentials are factored out, may themselves behave exponentially[5]. For example, in the expressions (38) for the corrections to the ground state energy, $E_0^{(1)}$ does behave like $e^{-b^2/4}$, but the appearance of the square of this factor in the expression for $E_0^{(2)}$ is misleading: an asymptotic expansion of the integral that multiplies it shows that $E_0^{(2)} \sim -2/b^2$, as $b \to \infty$. In the absence of any estimation for the higher order contributions, the possibility that we are actually dealing

[5] Note that in our results (36), (37), (38), the above-mentioned infinite sums appear transformed into integrals, preceded by the expected power of $e^{-b^2/4}$—the details of how this happens in the first-order case can be found in the appendix.

with an asymptotic series cannot be ruled out—the much weaker dependence of $E_0^{(2)}$ on b, compared to that of $E_0^{(1)}$, certainly does not provide evidence to the contrary. In that case, the approximation is optimized by truncating the series while it is still converging—one then obtains arbitrarily high precision as $b \to \infty$.

3.4. A two-state system

Consider a state that, at $t = 0$, is the sum of the symmetric and antisymmetric $n = 0$ states found above, for, say, $\ell = 0$. The spin-down component of this state is of order 1 while the spin-up component is of order ϵ. Roughly speaking, the particle is localized at $z = -b$ and its spin points along the negative z-axis. Then the standard two-state system analysis shows that the amplitudes to be in the spin-up and spin-down states at later times behave like $-\mathrm{i}\sin(\Omega t)$ and $\cos(\Omega t)$ respectively, where Ω's expansion in powers of ϵ starts with $\epsilon E_0^{(1)}/2$. The spin precesses in the tangent plane to the cylinder (i.e., perpendicularly to the magnetic field) while the particle oscillates from $-b$ to zero and back.

4. Concluding remarks

We have studied the problem of the motion of a spin-1/2 particle on a cylinder, in the presence of a radial magnetic field. A non-standard perturbative analysis, applicable to any perturbation of the harmonic oscillator, led to a recursion relation for the wavefunction and energy corrections, equations (30) and (31) respectively, with explicit results for the ground state in equations (38)–(41) and figures 3 and 4. It is worth emphasizing that the radial magnetic field of the problem can be approximated in the laboratory, as explained in detail in section 2.1.

As mentioned in the introduction, we have not been able to find a treatment of this problem in the literature. The motion of a spin-1/2 particle in the field of a magnetic monopole has been studied in detail, both in the non-relativistic [1, 7–9, 15] and relativistic [12] cases. Symmetry aspects of the problem have also been considered extensively (see, e.g., [11]), with the discovery of an underlying supersymmetry among the most notable results [2, 4, 10]. On the other hand, quantum *spinless* particles moving on curves or surfaces have been extensively studied (see, e.g., [3, 14, 16] and references therein) with a general discussion of the effects of a vector potential given in [6].

We end with a comment on the form of the unperturbed Hamiltonian used, equation (6). When dealing with the motion of a quantum particle on a surface, one can use a 3D Laplacian in the Hamiltonian and constrain the motion of the particle on the surface using a steep confining potential in the normal direction. It is well known that in this approach, which seems to be the one appropriate for practical applications, there is an induced potential for the motion along the surface, proportional to the square of the difference between the two principal curvatures of the surface (see, e.g., [3, 14] and references therein). In our case, this is a constant which only shifts the energy eigenvalues. Nevertheless, an obvious extension of our problem here would be the study of the motion on the surface of a slightly curved cylinder, in which case the above-mentioned induced potential would have to be taken into account.

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Appendix. First-order corrections to the ground state: the standard treatment

We outline here the standard first-order perturbative analysis of the problem, deriving the corrections to the ground state. Given that the zeroth-order spectrum is degenerate, we need to first diagonalize the interaction Hamiltonian in each degenerate subspace. One easily sees that the interaction only connects the pairs of eigenstates $|n, \ell, +\rangle$, $|n, \ell+1, -\rangle$. The appropriate zeroth-order basis is given by the *symmetric* and *antisymmetric* linear combinations

$$\begin{aligned} \left|n_\ell^{\mathrm{s}}\right\rangle \equiv \left|n_\ell^0\right\rangle &= \tfrac{1}{\sqrt{2}}(|n, \ell, +\rangle + |n, \ell+1, -\rangle) \\ \left|n_\ell^{\mathrm{a}}\right\rangle \equiv \left|n_\ell^1\right\rangle &= \tfrac{1}{\sqrt{2}}(-|n, \ell, +\rangle + |n, \ell+1, -\rangle). \end{aligned} \tag{A.1}$$

Note that we use the label ℓ for states that are equally localized at $-\ell b$ and $-(\ell+1)b$. The reason for renaming the states with numerical superscripts, instead of letters, will become apparent below. The expectation value of H_{int} in these states reproduces our result (38) for the first-order correction to the energy. It is interesting to see how the first-order correction to the wavefunction, conventionally given by an infinite sum, is brought into the closed form (40). The matrix elements of H_{int} in the above basis are

$$\begin{aligned} \left\langle n_\ell^0\right| H_{\mathrm{int}} \left|m_{\ell'}^0\right\rangle &= \epsilon \left\langle n_\ell \mid m_{\ell+1}\right\rangle \delta_{\ell\ell'}\delta_{\tilde{n}\tilde{m}} \\ \left\langle n_\ell^0\right| H_{\mathrm{int}} \left|m_{\ell'}^1\right\rangle &= \epsilon \left\langle n_\ell \mid m_{\ell+1}\right\rangle \delta_{\ell\ell'}\delta_{\tilde{n},\tilde{m}+1} \\ \left\langle n_\ell^1\right| H_{\mathrm{int}} \left|m_{\ell'}^1\right\rangle &= -\epsilon \left\langle n_\ell \mid m_{\ell+1}\right\rangle \delta_{\ell\ell'}\delta_{\tilde{n}\tilde{m}}, \end{aligned} \tag{A.2}$$

where $\langle n_\ell \mid m_{\ell+1}\rangle$ is the overlap between SHO eigenstates $|n\rangle$, $|m\rangle$, at a distance b apart and $\tilde{n}$ is the parity of n. Specifying to the symmetric ground state and taking $\ell = 0$, we find the first-order correction

$$\left|0_0^0\right\rangle^{(1)} = \sum_{k=1}^{\infty} \frac{\langle 0_0 \mid k_1\rangle}{k} \left|k_0^{\tilde{k}}\right\rangle. \tag{A.3}$$

We see that only states with $\ell = 0$ contribute. Furthermore, when k is even, only the symmetric state contributes while for k odd, only the antisymmetric one does. Using the fact that

$$\langle 0_0 \mid k_1\rangle = \frac{1}{\sqrt{k!}}\left(-\frac{b}{\sqrt{2}}\right)^k \mathrm{e}^{-\frac{b^2}{4}}, \tag{A.4}$$

we find for the spin-up component of the correction

$$\left|0_0^0, +\right\rangle^{(1)} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k\sqrt{k!}}\left(\frac{b}{\sqrt{2}}\right)^k \mathrm{e}^{-\frac{b^2}{4}} |k\rangle, \tag{A.5}$$

which implies

$$\begin{aligned} b\frac{\partial}{\partial b}\left(\mathrm{e}^{\frac{b^2}{4}}\left|0_0^0, +\right\rangle^{(1)}\right) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k!}}\left(\frac{b}{\sqrt{2}}\right)^k |k\rangle \\ &= \mathrm{e}^{\frac{b^2}{4}}(|0_1\rangle - |0_0\rangle). \end{aligned} \tag{A.6}$$

Integrating back w.r.t. b and changing integration variable we recover our earlier result (36) (similarly for the spin-down component).

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