Aproximación de funciones mediante polinomios de Bernstein

Definición 1. *Los* ***polinomios básicos de Bernstein*** *se definen para* $0 \leq k \leq n$ *y* $n = 1, ...$ *como*

$$\gamma_{n,k}(t) = \binom{n}{k} \ t^k \ (1-t)^{n-k}$$

donde

$$\binom{n}{k} = \frac{n!}{k! \ (n-k)!}$$

es el coeficiente binomial.

Ee la fórmula del binomio de Newton

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \ x^k \ y^{n-k}$$

Si tomamos $x = t$, $y = 1 - t$ tenemos

$$\sum_{k=0}^{n} \gamma_{n,k}(t) = 1 \tag{1}$$

Multiplicando la ecuación (1) por t para n-1 en vez de n, obtenemos

$$\begin{aligned}
t &= t \ \sum_{j=0}^{n-1} \gamma_{n-1,j}(t) \\
&= \sum_{j=0}^{n-1} \ \binom{n-1}{j} \ t^{j+1} \ (1-t)^{n-1-j} \\
&= \sum_{j=0}^{n-1} \ \left(\frac{j+1}{n}\right) \ \binom{n}{j+1} t^{j+1} \ (1-t)^{n-1-j} \\
&= \sum_{k=1}^{n-1} \ \left(\frac{k}{n}\right) \ \binom{n}{k} \ t^k \ (1-t)^{n-k} \\
&= \sum_{k=1}^{n-1} \ \left(\frac{k}{n}\right) \ \binom{n}{k} \ t^k \ (1-t)^{n-k} \\
&= \sum_{k=0}^{n} \ \left(\frac{k}{n}\right) \ \binom{n}{k} \ t^k \ (1-t)^{n-k} \\
&= \sum_{k=0}^{n} \ \left(\frac{k}{n}\right) \ \gamma_{n,k}(t)
\end{aligned}$$

Por lo tanto

$$\sum_{k=0}^{n} \ k \ \gamma_{n,k}(t) = nt \tag{2}$$

De manera análoga multiplicando la ecuación (1) por t^2 para n-2 en vez de n, obtenemos

$$
\begin{aligned}
t^2 &= t^2 \sum_{j=0}^{n-2} \gamma_{n-2,j}(t) \\
&= \sum_{j=0}^{n-2} \binom{n-2}{j} t^{j+2} (1-t)^{n-2-j} \\
&= \sum_{j=0}^{n-2} \left(\frac{(j+1)(j+2)}{n(n-1)}\right) \binom{n}{j+2} t^{j+2} (1-t)^{n-2-j} \\
&= \sum_{k=2}^{n} \left(\frac{k(k-1)}{n(n-1)}\right) \binom{n}{k} t^{k} (1-t)^{n-k} \\
&= \sum_{k=1}^{n} \left(\frac{k(k-1)}{n(n-1)}\right) \binom{n}{k} t^{k} (1-t)^{n-k} \\
&= \sum_{k=0}^{n} \left(\frac{k(k-1)}{n(n-1)}\right) \binom{n}{k} t^{k} (1-t)^{n-k} \\
&= \sum_{k=0}^{n} \left(\frac{k(k-1)}{n(n-1)}\right) \gamma_{n,k}(t)
\end{aligned}
$$

Por lo tanto

$$\sum_{k=0}^{n} (k^2-k)\ \gamma_{n,k}(t) = (n^2-n)t^2 \tag{3}$$

Lema 1. *Para todo $n \in \mathbb{N}$ se cumple*

$$\sum_{k=0}^{n} \left(t - \frac{k}{n}\right)^2 \gamma_{n,k}(t) \leq \frac{1}{4n}$$

Demostración. Multiplicando la igualdad (1) por t^2 se obtiene

$$\sum_{k=0}^{n} t^2\ \gamma_{n,k}(t) = t^2 \tag{4}$$

Multiplicando la igualdad (2) por $-\frac{2}{n}t$ se obtiene

$$\sum_{k=0}^{n} -2\ \frac{k}{n}\ t\ \ \gamma_{n,k}(t) = -2t^2 \tag{5}$$

Sumando las igualdad (2) y (3) se obtiene

$$
\begin{aligned}
\sum_{k=0}^{n} k\ \gamma_{n,k}(t) + \sum_{k=0}^{n} (k^2-k)\ \gamma_{n,k}(t) &= nt + (n^2-n)t^2 \\
\sum_{k=0}^{n} \frac{k^2}{n^2}\ \gamma_{n,k}(t) &= \left(1 - \frac{1}{n}\right)\ t^2 + \frac{1}{n}t
\end{aligned}
$$

Por lo tanto

$$\sum_{k=0}^{n} \frac{k^2}{n^2}\ \gamma_{n,k}(t) = \left(1 - \frac{1}{n}\right)\ t^2 + \frac{1}{n} t \tag{6}$$

Sumando las igualdades (4), (5) y (6) se tiene

$$\sum_{k=0}^{n} t^2\ \gamma_{n,k}(t) + \sum_{k=0}^{n}\ -2\ \frac{k}{n}\ t\ \ \gamma_{n,k}(t) + \sum_{k=0}^{n} \frac{k^2}{n^2}\ \gamma_{n,k}(t) = t^2 + -2t^2 + \left(1 - \frac{1}{n}\right)\ t^2 + \frac{1}{n} t$$

por lo tanto

$$\sum_{k=0}^{n} \left(t - \frac{k}{n}\right)^2\ \gamma_{n,k}(t) = \frac{1}{n}(t - t^2)$$

y dado que $\max\limits_{t\in[0,1]} (t - t^2) = \dfrac{1}{4}$ tenemos que

$$\sum_{k=0}^{n} \left(t - \frac{k}{n}\right)^2\ \gamma_{n,k}(t) \leq \frac{1}{4n} \tag{7}$$

□

Definición 2. *Sea* $f : [0,1] \to \mathbb{R}$ *una función continua. El* ***n-ésimo polinomio de Bernstein*** *de* f *es el polinomio*

$$\beta_n(t) = \beta_{f,n}(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\ t^k\ (1-t)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\gamma_{n,k}(t)$$

Teorema 1. *Sea* $f : [0,1] \to \mathbb{R}$ *una función continua. La sucesión de polinomios de Bernstein* $(\beta_{f,n})$ *converge uniformemente a* f *en* $[0,1]$

Demostración. Sea $\epsilon > 0$. Como f es uniformemente continua, existe $\delta > 0$ tal que

$$|f(s) - f(t)| < \frac{\epsilon}{2}\ \ si\ \ |s - t| < \delta \tag{8}$$

Fijemos $t \in [0,1]$. Multiplicando por $f(t)$ la igualdad 1 obtenemos que

$$f(t) = \sum_{k=0}^{n} f(t)\ \gamma_{n,k}(t)$$

En consecuencia

$$|f(t) - \beta_{f,n}(t)| = \left|\sum_{k=0}^{n} f(t)\ \gamma_{n,k}(t) - \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\ \gamma_{n,k}(t)\right| \leq \sum_{k=0}^{n}\left|f(t) - f\left(\frac{k}{n}\right)\right|\ \gamma_{n,k}(t) \tag{9}$$

Probaremos que el lado derecho de esta desigualdad es menor que ϵ si n satisface

$$n \geq \max\left\{\frac{1}{\delta^4}, \frac{\|f\|_\infty^2}{\epsilon^2}\right\}$$

Para ello consideremos los conjuntos

$$I_1 = \left\{ k \in \mathbb{N} \cup \{0\} \mid 0 \leq k \leq n, \ \ \left| \frac{k}{n} - t \right| < \left(\frac{1}{n} \right)^{\frac{1}{4}} \right\}$$

$$I_2 = \{ k \in \mathbb{N} \cup \{0\} \mid 0 \leq k \leq n, \ k \notin I_1 \}$$

Para $k \in I_1$ supongamos que

$$n \geq \frac{1}{\delta^4} \ \Rightarrow \ \delta^4 \geq \frac{1}{n} \ \Rightarrow \ \delta \geq \left(\frac{1}{n} \right)^{\frac{1}{4}}$$

por tanto por continuidad uniforme

$$\begin{aligned} \sum_{k \in I_1}^{n} \left| f(t) - f\left(\frac{k}{n} \right) \right| \ \gamma_{n,k}(t) &\leq \sum_{k \in I_1}^{n} \frac{\epsilon}{2} \ \gamma_{n,k}(t) \\ &= \frac{\epsilon}{2} \sum_{k \in I_1}^{n} \ \ \gamma_{n,k}(t) \\ &= \frac{\epsilon}{2}(1) \\ &= \frac{\epsilon}{2} \end{aligned}$$

Para $k \in I_2$ supongamos que

$$\left| \frac{k}{n} - t \right| \geq \left(\frac{1}{n} \right)^{\frac{1}{4}} \ \Rightarrow \ n^{\frac{1}{4}} \geq \frac{1}{\left| \frac{k}{n} - t \right|} \ \Rightarrow \ n^{\frac{1}{2}} \geq \frac{1}{\left| \frac{k}{n} - t \right|^2}$$

y si

$$n \geq \text{máx} \left\{ \frac{1}{\delta^4}, \frac{\|f\|_\infty^2}{\epsilon^2} \right\}$$

considerando que

$$n \geq \frac{\|f\|_\infty^2}{\epsilon^2} \ \Rightarrow \ n^{\frac{1}{2}} \geq \frac{\|f\|_\infty}{\epsilon} \ \Rightarrow \ \epsilon \geq \frac{\|f\|_\infty}{n^{\frac{1}{2}}}$$

entonces

$$
\begin{aligned}
\sum_{k\in I_2}^{n} \left| f(t) - f\left(\frac{k}{n}\right) \right| \ \gamma_{n,k}(t) &\leq \sum_{k\in I_2}^{n} \ \|f\|_\infty \ \gamma_{n,k}(t) \\
&\leq \sum_{k\in I_2}^{n} \ 2\|f\|_\infty \ \gamma_{n,k}(t) \\
&= 2\|f\|_\infty \ \sum_{k\in I_2}^{n} \ \frac{\left|\frac{k}{n} - t\right|^2}{\left|\frac{k}{n} - t\right|^2} \ \gamma_{n,k}(t) \\
&\leq 2\|f\|_\infty \ \ n^{\frac{1}{2}} \ \sum_{k\in I_2}^{n} \ \left|\frac{k}{n} - t\right|^2 \ \gamma_{n,k}(t) \\
&= 2\|f\|_\infty \ \frac{n}{n^{\frac{1}{2}}} \ \sum_{k\in I_2}^{n} \ \left|\frac{k}{n} - t\right|^2 \ \gamma_{n,k}(t) \\
&< 2 \ n \ \epsilon \ \frac{1}{4n} \\
&= \frac{\epsilon}{2}
\end{aligned}
$$

Por lo tanto si si n satisface

$$
n \geq \max\left\{\frac{1}{\delta^4}, \frac{\|f\|_\infty^2}{\epsilon^2}\right\}
$$

entonces

$$
\begin{aligned}
|f(t) - \beta_{f,n}(t)| &\leq \sum_{k=0}^{n} \ \left| f(t) - f\left(\frac{k}{n}\right) \right| \ \gamma_{n,k}(t) \\
&= \sum_{k\in I_1}^{n} \ \left| f(t) - f\left(\frac{k}{n}\right) \right| \ \gamma_{n,k}(t) + \sum_{k\in I_2}^{n} \ \left| f(t) - f\left(\frac{k}{n}\right) \right| \ \gamma_{n,k}(t) \\
&< \epsilon
\end{aligned}
$$

para toda $t \in [0,1]$ es decir

$$
\|f - \beta_{f,n}\|_\infty < \epsilon \ \ si \ \ n \geq \max\left\{\frac{1}{\delta^4}, \frac{\|f\|_\infty^2}{\epsilon^2}\right\}
$$

$\square$