Polinomios de Bernstein

Definición 1. *Los polinomios básicos de Bernstein se definen para $0 \leq k \leq n$ y $n = 1, \ldots$ como*

$$\gamma_{n,k}(t) = \binom{n}{k} \; t^k \; (1-t)^{n-k}$$

donde

$$\binom{n}{k} = \frac{n!}{k! \; (n-k)!}$$

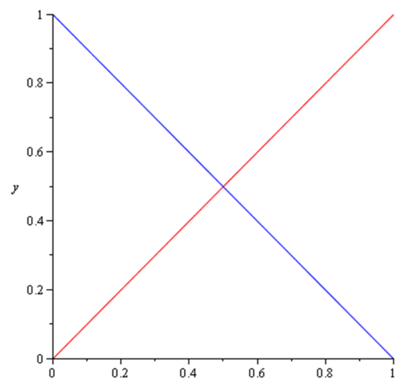
es el coeficiente binomial.

Ejemplos Tenemos que

1. Los polinomios básicos de Bernstein de grado 1

$$\gamma_{1,0}(t) = \binom{1}{0} \; t^0 \; (1-t)^{1-0} = 1 - t$$

$$\gamma_{1,1}(t) = \binom{1}{1} \; t^1 \; (1-t)^{1-1} = t$$

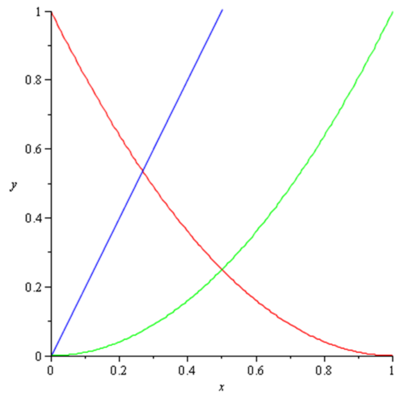


2. Los polinomios básicos de Bernstein de grado 2

$$\gamma_{2,0}(t) = \binom{2}{0} \; t^0 \; (1-t)^{2-0} = (1-t)^2$$

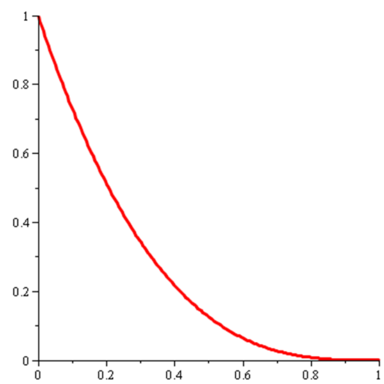
$$\gamma_{2,1}(t) = \binom{2}{1} \; t^1 \; (1-t)^{2-1} = 2t(1-t)$$

$$\gamma_{2,2}(t) = \binom{2}{1} \; t^2 \; (1-t)^{2-2} = t^2$$



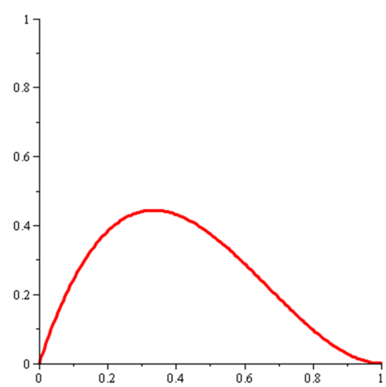
3. Los polinomios básicos de Bernstein de grado 3

$$\gamma_{3,0}(t) = \binom{3}{0}\ t^0\ (1-t)^{3-0} = (1-t)^3$$



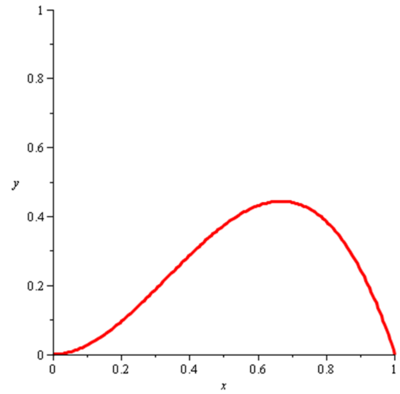
4.

$$\gamma_{3,1}(t) = \binom{3}{1}\ t^1\ (1-t)^{3-1} = 3t(1-t)^2$$



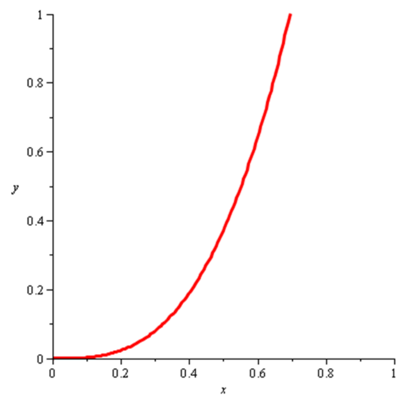
5.

$$\gamma_{3,2}(t) = \binom{3}{2} t^2 \ (1-t)^{3-2} = 3t^2(1-t)$$



6.

$$\gamma_{3,3}(t) = \binom{3}{3} t^3 \ (1-t)^{3-3} = t^3$$



Aproximación de funciones mediante Polinomios de Bernstein

A continuación estudiaremos el comportamiento de los polinomios de Bernstein cuando aproximamos funciones con ellos.
Ahora bien de la fórmula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \ x^k \ y^{n-k}$$

Si tomamos $x = t$, $y = 1-t$ tenemos

$$\begin{aligned} 1 &= \sum_{k=0}^{n} \binom{n}{k} \ t^k \ (1-t)^{n-k} \\ &= \sum_{k=0}^{n} \gamma_{n,k}(t) \end{aligned}$$

Ahora bien de la fórmula

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} \ x^k \ y^{n-k}$$

Derivando con respecto a x se tiene

$$n \ (x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \ k \ x^{k-1} \ y^{n-k}$$

multiplicando por $\dfrac{x}{n}$ se tiene

$$x \ (x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \ \frac{k}{n} \ x^k \ y^{n-k}$$

Si tomamos $x = t$, $y = 1-t$ tenemos

$$\begin{aligned} t &= \sum_{k=0}^{n} \binom{n}{k} \ \left(\frac{k}{n}\right) \ t^k \ (1-t)^{n-k} \\ &= \sum_{k=0}^{n} \left(\frac{k}{n}\right) \gamma_{n,k}(t) \end{aligned}$$

Ahora bien de la fórmula

$$x \ (x+y)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} \ \frac{k}{n} \ x^k \ y^{n-k}$$

Derivando con respecto a x se tiene

$$\begin{aligned} x(n-1)\ (x+y)^{n-2}+(x+y)^{n-1} &= \sum_{k=0}^{n}\binom{n}{k}\ \frac{k^2}{n}\ x^{k-1}\ y^{n-k} \\ xn(x+y)^{n-2}-x(x+y)^{n-2}+(x+y)^{n-1} &= \sum_{k=0}^{n}\binom{n}{k}\ \frac{k^2}{n}\ x^{k-1}\ y^{n-k} \\ xn(x+y)^{n-2}+y(x+y)^{n-2} &= \sum_{k=0}^{n}\binom{n}{k}\ \frac{k^2}{n}\ x^{k-1}\ y^{n-k} \end{aligned}$$

multiplicando por $\frac{x}{n}$ se tiene

$$x(x+y)^{n-2}+\frac{xy}{n}(x+y)^{n-2}=\sum_{k=0}^{n}\binom{n}{k}\ \frac{k^2}{n^2}\ x^{k}\ y^{n-k}$$

Si tomamos $x=t$, $y=1-t$ tenemos

$$\begin{aligned} t^2+\frac{t(1-t)}{n} &= \sum_{k=0}^{n}\binom{n}{k}\ \left(\frac{k}{n}\right)^2\ t^k\ (1-t)^{n-k} \\ &= \sum_{k=0}^{n}\left(\frac{k}{n}\right)^2\gamma_{n,k}(t) \end{aligned}$$

Definición 2. *Sea $f:[0,1]\to\mathbb{R}$ una función continua. El* ***n-ésimo polinomio de Bernstein*** *de f es el polinomio*

$$\beta_n(t)=\beta_{f,n}(t)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}\ t^k\ (1-t)^{n-k}=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\gamma_{n,k}(t)$$

Ejemplo Para las funciones $f(t)=1$, $f(t)=t$ y $f(t)=t^2$ se tiene

1. $f(t)=1$

$$\beta_n(t)=\beta_{1,n}(t)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\gamma_{n,k}(t)=\sum_{k=0}^{n}(1)\gamma_{n,k}(t)=1$$

   En este caso

$$|f(t)-\beta_{1,n}(t)|=|1-1|=0<\epsilon$$

2. $f(t)=t$

$$\beta_n(t)=\beta_{t,n}(t)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\gamma_{n,k}(t)\sum_{k=0}^{n}\left(\frac{k}{n}\right)\gamma_{n,k}(t)=t$$

   En este caso

$$|f(t)-\beta_{t,n}(t)|=|t-t|=0<\epsilon$$

3. $f(t) = t^2$

$$\beta_n(t) = \beta_{t^2,n}(t) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\gamma_{n,k}(t)\sum_{k=0}^{n}\left(\frac{k}{n}\right)^2\gamma_{n,k}(t) = t^2 + \frac{t(1-t)}{n}$$

En este caso

$$|f(t) - \beta_{t^2,n}(t)| = \left|t^2 - \left(t^2 + \frac{t(1-t)}{n}\right)\right| = \left|-\left(\frac{t(1-t)}{n}\right)\right| \underbrace{\leq}_{0\leq t\leq 1} \frac{1}{4n}$$

donde $\dfrac{1}{4n} \to 0$ cuando $n \to \infty$

Las buenas propiedades de los polinomios de Bernstein nos da esperanzas de que $\beta_{f,n}(t)$ podría ser una buena aproximación para $f(t)$ sobre $[0,1]$.