Integrales Elipticas

Longitud de una Curva

Sea f una función continua en $[a, b]$. Si $\{t_0, t_1, ..., t_n\}$ es una partición de $[a, b]$ tenemos que en el intervalo $[t_{i-1}, t_i]$ al aplicar f, la distancia entre ellos es:

$$d((x_{i-1}, f(x_{i-1})), (x_i, f(x_i))) = \sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2}$$

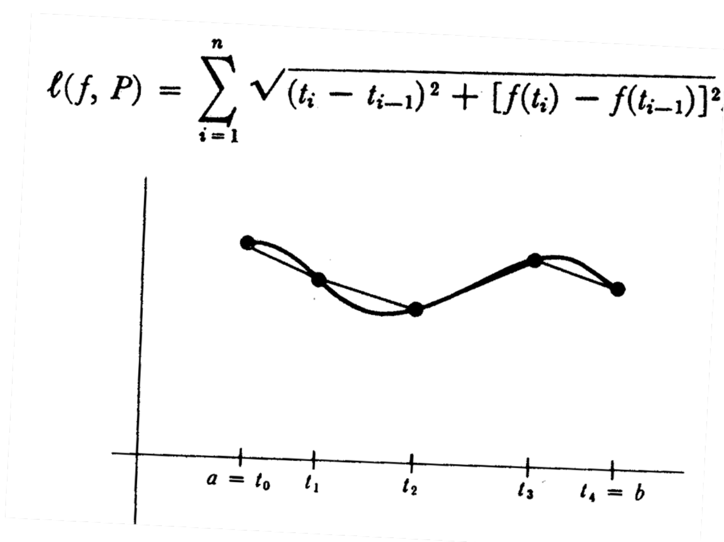
Ahora bien si aplicamos el Teorema del Valor medio al intervalo (t_{i-1}, t_i) se tiene que existe $c \in (t_{i-1}t_i)$ tal que

$$\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} = f'(c) \quad \Rightarrow \quad f(t_i) - f(t_{i-1}) = f'(c)((t_i - t_{i-1})$$

se tiene entonces que

$$\sqrt{(t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2} = \sqrt{(t_i - t_{i-1})^2 + (f'(c)((t_i - t_{i-1}))^2} = (t_i - t_{i-1})\sqrt{1 + (f'(c))^2}$$

este proceso aplicable a la partición P



nos da una aproximación a la longitud de la grafica de la función, esto es

$$\ell(f, P) \approx \sum_{i=1}^{n} (t_i - t_{i-1})\sqrt{1 + (f'(c_i))^2} \qquad c_i \in (x_{i-1}, x_i)$$

por lo tanto

$$\ell(f, P) = \lim_{n \to \infty} \sum_{i=1}^{n} (t_i - t_{i-1})\sqrt{1 + (f'(c_i))^2} \qquad c_i \in (x_{i-1}, x_i)$$

este término corresponde a una suma de Riemann, por lo tanto

$$\ell(f,P)=\int_a^b\sqrt{1+(f'(x))^2}dx$$

Longitud de la Elipse

Tenemos que dada la elipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ \Rightarrow\ y=b\sqrt{1-\frac{x^2}{a^2}}\ \Rightarrow\ y'=-\frac{bx}{\sqrt{\frac{a^2-x^2}{a^2}}a^2}$$

Por lo que su logitud en un intervalo de $[0,a]$ sera:

$$\int_0^a\sqrt{1+\left(-\frac{bx}{\sqrt{\frac{a^2-x^2}{a^2}}a^2}\right)^2}dt=\frac{1}{a}\int_0^a\sqrt{\frac{a^4-a^2x^2+b^2x^2}{(a^2-x^2)}}dx$$

$$\underbrace{=}_{\substack{x=a\operatorname{sen}\ t\\ dx=a\cos\ t\ dt}}\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{\frac{a^4-a^2a^2\operatorname{sen}^2\ t+b^2a^2\operatorname{sen}^2 t}{(a^2-a^2\operatorname{sen}^2 t)}}(a\cos\ t)dt=\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{\frac{a^4-a^2(a^2-b^2)\operatorname{sen}^2\ t}{a^2(1-\operatorname{sen}^2 t)}}(a\cos\ t)dt$$

$$=\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{\frac{a^4-a^2(a^2-b^2)\operatorname{sen}^2\ t}{a^2\cos^2 t}}(a\cos\ t)dt=\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^4-a^2(a^2-b^2)\operatorname{sen}^2\ t}dt$$

$$\underbrace{=}_{\substack{a^2=b^2+c^2\ \Rightarrow\ e=\frac{c}{a}=\frac{\sqrt{a^2-b^2}}{a}\\ \therefore\ e^2=\frac{a^2-b^2}{a^2}\ \Rightarrow\ a^2e^2=a^2-b^2}}\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^4-a^2(a^2e^2)\operatorname{sen}^2\ t}\ dt=\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^4-a^4e^2\operatorname{sen}^2\ t}\ dt$$

$$=\frac{1}{a}\int_0^{\frac{\pi}{2}}\sqrt{a^4(1-e^2\operatorname{sen}^2\ t)}\ dt=\frac{a^2}{a}\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\operatorname{sen}^2\ t}\ dt=a\int_0^{\frac{\pi}{2}}\sqrt{1-e^2\operatorname{sen}^2\ t}\ dt$$

Si hacemos $k^2=e^2$ se obtiene

$$a\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\operatorname{sen}^2(t)}dt$$

Definición 1. *A las integrales*

$$\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\operatorname{sen}^2(t)}dt\quad 0<k<1$$

se les conoce como integral eliptica de segunda especie

Vamos a ver un método para aproximar las integrales elipticas de segunda especie dado el binomio de Newton

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4 + \cdots$$

tenemos que

$$(1-x)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \frac{1}{2}1^{\frac{1}{2}-1}(-x) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}1^{\frac{1}{2}-2}(-x)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}1^{\frac{1}{2}-3}(-x)^3 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)(\frac{1}{2}-3)}{4!}1^{n-4}(-x)^4 + \cdots$$

$$= 1 - \frac{x}{2} - \left(\frac{1}{2}\right)\frac{x^2}{4} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)\frac{x^3}{6} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{x^4}{8} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\frac{x^5}{10} - \cdots$$

por lo tanto

$$\sqrt{1-x} = 1 - \frac{x}{2} - \left(\frac{1}{2}\right)\frac{x^2}{4} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)\frac{x^3}{6} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{x^4}{8} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\frac{x^5}{10} - \cdots$$

si hacemos $x = k^2\operatorname{sen}^2(t)$ se obtiene

$$\sqrt{1-k^2\operatorname{sen}^2(t)} = 1 - \frac{k^2\operatorname{sen}^2(t)}{2} - \left(\frac{1}{2}\right)\frac{(k^2\operatorname{sen}^2(t))^2}{4} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)\frac{(k^2\operatorname{sen}^2(t))^3}{6} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{(k^2\operatorname{sen}^2(t))^4}{8} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\frac{(k^2\operatorname{sen}^2(t))^5}{10} - \cdots$$

por lo tanto

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\operatorname{sen}^2(t)}dt = \int_0^{\frac{\pi}{2}} 1 - \frac{k^2\operatorname{sen}^2(t)}{2} - \left(\frac{1}{2}\right)\frac{(k^2\operatorname{sen}^2(t))^2}{4} - \left(\frac{1}{2}\cdot\frac{3}{4}\right)\frac{(k^2\operatorname{sen}^2(t))^3}{6} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{(k^2\operatorname{sen}^2(t))^4}{8} - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\frac{(k^2\operatorname{sen}^2(t))^5}{10} - \cdots$$

$$= \int_0^{\frac{\pi}{2}} 1dt - \int_0^{\frac{\pi}{2}} \frac{k^2\operatorname{sen}^2(t)}{2}dt - \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\right)\frac{(k^2\operatorname{sen}^2(t))^2}{4}dt - \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\cdot\frac{3}{4}\right)\frac{(k^2\operatorname{sen}^2(t))^3}{6}dt - \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)\frac{(k^2\operatorname{sen}^2(t))^4}{8}dt - \int_0^{\frac{\pi}{2}} \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)\frac{(k^2\operatorname{sen}^2(t))^5}{10}dt - \cdots$$

vamos a calcular por separado el valor de cada integral

$$\int_0^{\frac{\pi}{2}} 1dt = \frac{\pi}{2}, \quad \int_0^{\frac{\pi}{2}} k^2\operatorname{sen}^2(t)dt = \frac{k^2\pi}{4}, \quad \int_0^{\frac{\pi}{2}} (k^2\operatorname{sen}^2(t))^2dt = \frac{3}{16}\pi k^4, \quad \int_0^{\frac{\pi}{2}} (k^2\operatorname{sen}^2(t))^3dt = \frac{5}{32}\pi k^6$$

$$\int_0^{\frac{\pi}{2}} (k^2\operatorname{sen}^2(t))^4dt = \frac{35}{256}\pi k^8, \quad \int_0^{\frac{\pi}{2}} (k^2\operatorname{sen}^2(t))^5dt = \frac{63}{512}\pi k^{10}$$

por lo tanto

$$\int_0^{\frac{\pi}{2}} \sqrt{1-k^2\operatorname{sen}^2(t)}dt = \frac{\pi}{2} - \left(\frac{1}{2}\right)\frac{k^2\pi}{4} - \left(\frac{1}{2}\cdot\frac{1}{4}\right)\frac{3}{16}\pi k^4 - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{6}\right)\frac{5}{32}\pi k^6 - \left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{8}\right)\frac{35}{256}\pi k^8$$

$$-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{1}{10}\right)\frac{63}{512}\pi k^{10}-\cdots$$

$$=\left(\frac{\pi}{2}\right)\left(1-\left(\frac{1}{2}\right)^2k^2-\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\frac{k^4}{3}-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\frac{k^6}{5}-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\frac{k^8}{7}-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\right)^2\frac{k^{10}}{9}\right)$$

Ejemplo Calcula la longitud de la gráfica de la función $\operatorname{sen}(x)$ en $\left[0,\frac{\pi}{2}\right]$

Solución tenemos que según la fórmula

$$\ell(f,P)=\int_a^b\sqrt{1+(f'(x))^2}dx$$

se tiene

$$\int_0^{\frac{\pi}{2}}\sqrt{1+(\cos(x))^2}dx=\int_0^{\frac{\pi}{2}}\sqrt{1+(1-\operatorname{sen}^2(x))}dx=\int_0^{\frac{\pi}{2}}\sqrt{2-\operatorname{sen}^2(x)}dx=\int_0^{\frac{\pi}{2}}\sqrt{2\left(1-\frac{1}{2}\operatorname{sen}^2(x)\right)}dx$$

$$=\sqrt{2}\int_0^{\frac{\pi}{2}}\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\operatorname{sen}^2(x)}dx\quad en\quad este\quad caso\quad k=\frac{1}{\sqrt{2}}$$

y usamos nuestra fórmula y tenemos que

$$\int_0^{\frac{\pi}{2}}\sqrt{1+(\cos(x))^2}dx=\sqrt{2}\left(\frac{\pi}{2}\right)\left(1-\left(\frac{1}{2}\right)^2\left(\frac{1}{\sqrt{2}}\right)^2-\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\frac{\left(\frac{1}{\sqrt{2}}\right)^4}{3}-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\frac{\left(\frac{1}{\sqrt{2}}\right)^6}{5}-\right.$$

$$\left.\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2\frac{\left(\frac{1}{\sqrt{2}}\right)^8}{7}-\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\right)^2\frac{\left(\frac{1}{\sqrt{2}}\right)^{10}}{9}\right)\approx 1{,}910355219$$

Longitud de la Lemniscata

Dada una ecuación de una curva en coordenadas polares

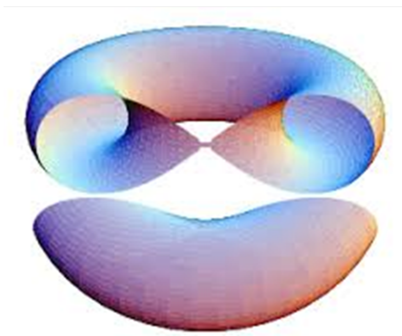
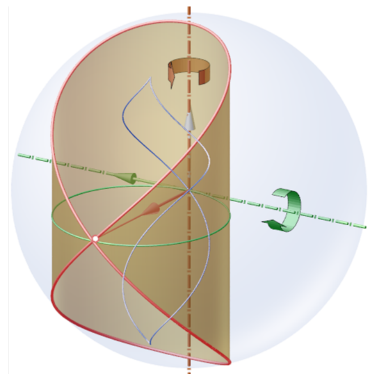
$$(f(\theta)\cos\ \theta, f(\theta)\operatorname{sen}\ \theta)$$

Tenemos que su longitud será

$$\int_a^b \sqrt{([f(\theta)\cos\ \theta]')^2 + ([f(\theta)\operatorname{sen}\ \theta]')^2}d\theta = \int_a^b \sqrt{f^2(\theta) + f'^2(\theta)}\ d\theta$$

Ahora bien dada la ecuación de la lemniscata

$$r^2 = 2a^2\cos\ 2t$$

Tenemos que su longitud será

$$\int_a^b \sqrt{2a^2\cos\ 2t + \frac{2a^2\operatorname{sen}^2 2t}{\cos\ 2t}}\ dt = \int_a^b \sqrt{\frac{2a^2\cos^2 2t + 2a^2\operatorname{sen}^2 2t}{\cos\ 2t}}\ dt = \sqrt{2}a\int_a^b \frac{dt}{\sqrt{\cos\ 2t}}$$

de donde

$$\sqrt{2}a\int_a^b \frac{dt}{\sqrt{\cos\ 2t}} = \sqrt{2}a\int_a^b \frac{dt}{\sqrt{1-2\operatorname{sen}^2\ t}}$$

Definición 2. *A las integrales*

$$\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-k^2\operatorname{sen}^2(t)}} \quad 0<k<1$$

se les conoce como integral eliptica de primer especie

Vamos a mostrar un método para aproximar estas integrales

Ejercicio Demuestra que si $0 < k < 1$

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\operatorname{sen}^2(t)}}dt = \left(\frac{\pi}{2}\right)\left(1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2k^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2k^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2k^8+\right.$$

$$\left.\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\right)^2k^{10}\right)$$

Demostración. dado el binomio de Newton

$$(a+b)^n = a^n+na^{n-1}b+\frac{n(n-1)}{2!}a^{n-2}b^2+\frac{n(n-1)(n-2)}{3!}a^{n-3}b^3+\frac{n(n-1)(n-2)(n-3)}{4!}a^{n-4}b^4+\cdots$$

tenemos que

$$(1-x)^{-\frac{1}{2}} = 1^{-\frac{1}{2}}+\left(-\frac{1}{2}\right)1^{-\frac{1}{2}-1}(-x)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right))}{2!}1^{-\frac{1}{2}-2}(-x)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}1^{-\frac{1}{2}-3}(-x)^3+$$

$$\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!}1^{-\frac{1}{2}-4}(-x)^4+\cdots$$

$$= 1+\frac{x}{2}+\left(\frac{1}{2}\cdot\frac{3}{4}\right)x^2+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)x^3+\cdots$$

por lo tanto si hacemos $x = k^2\operatorname{sen}^2(t)$ se obtiene

$$\left(1-k^2\operatorname{sen}^2(t)\right)^{-\frac{1}{2}} = 1+\frac{k^2\operatorname{sen}^2(t)}{2}+\left(\frac{1}{2}\cdot\frac{3}{4}\right)(k^2\operatorname{sen}^2(t))^2+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)(k^2\operatorname{sen}^2(t))^3+\cdots$$

por lo tanto

$$\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\operatorname{sen}^2(t)}}dt = \int_0^{\frac{\pi}{2}}\left(1+\frac{k^2\operatorname{sen}^2(t)}{2}+\left(\frac{1}{2}\cdot\frac{3}{4}\right)(k^2\operatorname{sen}^2(t))^2+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)(k^2\operatorname{sen}^2(t))^3+\cdots\right)$$

$$= \left(\frac{\pi}{2}\right)\left(1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2k^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2k^6+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\right)^2k^8+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{7}{8}\cdot\frac{9}{10}\right)^2k^{10}\right)$$

□

Ejemplo Calcular $\displaystyle\int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}}$

Solución tenemos que haciendo el cambio $x = 2\operatorname{sen}(t)$ obtenemos

$$\int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}} \underbrace{=}_{\substack{x=2\operatorname{sen}(t)\\ dx=2\cos(t)dt}} \int_0^{\frac{\pi}{2}} \frac{2\cos(t)dt}{\sqrt{(4-4\operatorname{sen}^2(t))(9-4\operatorname{sen}^2(t))}} = \int_0^{\frac{\pi}{2}} \frac{2\cos(t)dt}{\sqrt{(4(1-\operatorname{sen}^2(t))(9(1-\frac{4}{9}\operatorname{sen}^2(t))}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cancel{2}\cos(t)dt}{\cancel{2}\sqrt{(1-\operatorname{sen}^2(t)(9(1-\frac{4}{9}\operatorname{sen}^2(t))}} = \int_0^{\frac{\pi}{2}} \frac{\cos(t)dt}{\sqrt{\cos^2(t)}\sqrt{9(1-\frac{4}{9}\operatorname{sen}^2(t))}} = \left(\frac{1}{3}\right)\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\frac{4}{9}\operatorname{sen}^2(t)}}$$

en este caso $k^2 = \dfrac{4}{9} \quad \Rightarrow \quad k = \dfrac{2}{3}$ y usando nuestra fórmula

$$\left(\frac{1}{3}\right)\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{1-\frac{4}{9}\operatorname{sen}^2(t)}} = \left(\frac{1}{3}\right)\left(\frac{\pi}{2}\right)\left(1+\left(\frac{1}{2}\right)^2\left(\frac{2}{3}\right)^2+\left(\frac{1}{2}\cdot\frac{3}{4}\right)^2\left(\frac{2}{3}\right)^4+\left(\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\right)^2\left(\frac{2}{3}\right)^6\right) \approx 0{,}600809843$$