Diferenciales de funciones $f : A \subset \mathbb{R}^2 \to \mathbb{R}$

Tenemos que $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ es diferenciable si

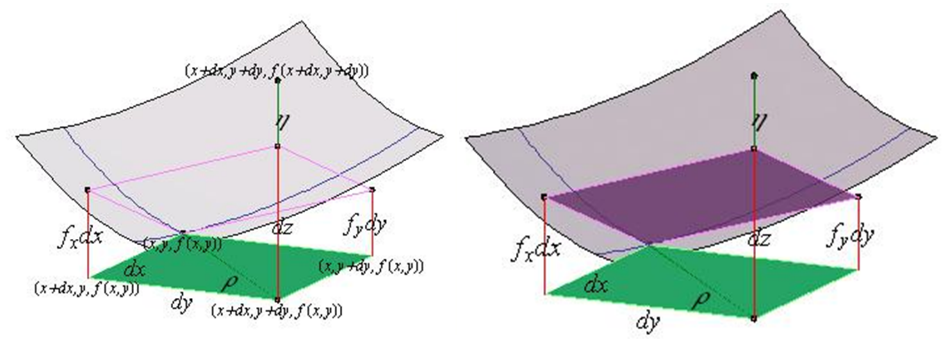
$$f(x_o + h_1, y_0 + h_2) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2 + r(h_1, h_2)$$

cumple

$$\lim_{(h_1,h_2)\to(0,0)} \frac{r(h_1, h_2)}{\|(h_1, h_2)\|} = 0$$

Esto se puede escribir como

$$f(x_o + h_1, y_0 + h_2) - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2 + r(h_1, h_2)$$



tomando

$$f(x_o + h_1, y_0 + h_2) - f(x_0, y_0) = \triangle z$$
$$\frac{\partial f}{\partial x}(x_0, y_0)h_1 = \frac{\partial f}{\partial x}(x_0, y_0)\triangle x$$
$$\frac{\partial f}{\partial y}(x_0, y_0)h_2 = \frac{\partial f}{\partial y}(x_0, y_0)\triangle y$$

tenemos que

$$\triangle z = \frac{\partial f}{\partial x}(x_0, y_0)\triangle x + \frac{\partial f}{\partial y}(x_0, y_0)\triangle y + r(\triangle x, \triangle y)$$

haciendo $\triangle x,\ \triangle y \to 0$ tenemos

$$dz = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy$$

Definición 1. *Si $z = f(x, y)$ es una función diferenciable, la diferencial de f denotada dz se define*

$$dz = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy$$

Ejemplo Calcular la diferencial de $z = 4x^2 - xy$

En este caso

$$dz = \frac{\partial(4x^2 - xy)}{\partial x}dx + \frac{\partial(4x^2 - xy)}{\partial y}dy = (8x - y)dx - xdy$$

Ahora bien

$$f(x_o + h_1, y_0 + h_2) - f(x_0, y_0) = \triangle z \approx \frac{\partial f}{\partial x}(x_0, y_0)\triangle x + \frac{\partial f}{\partial y}(x_0, y_0)\triangle y$$

expresa el cambio aproximado de $z = f(x, y)$ cuando (x, y) pasa a $(x + \triangle x, y + \triangle y)$

Ejemplo Aproximar el cambio de $z = 4x^2 - xy$ cuando (x, y) pasa de $(2, 1)$ a $(2{,}1, 1{,}5)$

En este caso tomamos $x_0 = 2$, $y_0 = 1$, $\triangle x = 0{,}1$ y $\triangle y = ,5$ y el valor de cambio será

$$\frac{\partial f}{\partial x}(2, 1)\triangle x + \frac{\partial f}{\partial y}(2, 1)\triangle y = (15)(0{,}1) - 2(0{,}5) = 1{,}5$$

mientras que

$$f(2{,}1, 1{,}5) - f(2, 1) = 14{,}49 - 14 = 0{,}49$$

por lo tanto en la aproximacion se cometio un error de 0,01

Definición 2. *Si dz es el error de medición en una cantidad z, el error relativo se define* $\dfrac{dz}{z}$

Ejemplo Demostrar que el error relativo en un producto $z = xy$ es la suma de los errores relativos de los factores

Solución En este caso

$$dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = y\ dx + x\ dy$$

por lo tanto

$$\frac{dz}{z} = \frac{y\ dx + x\ dy}{xy} = \frac{dx}{x} + \frac{dy}{y}$$

Diferencial de orden 2

Si $df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy$ entonces una diferencial de orden 2 seria:

$$d^2 f = d(df) = d\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)dy$$

$$= \left(\frac{\partial^2 f}{\partial x^2}dx + \frac{\partial^2 f}{\partial x \partial y}dy\right)dx + \left(\frac{\partial^2 f}{\partial y \partial x}dx + \frac{\partial^2 f}{\partial y^2}dy\right)dy = \frac{\partial^2 f}{\partial x^2}dx^2 + \frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y \partial x}dydx + \frac{\partial^2 f}{\partial y^2}dy^2$$

$$= \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

Por lo tanto

$$d^2 f = d(df) = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

Diferencial de orden 3

Si $d^2f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$ entonces una diferencial de orden 3 seria:

$$d^3f = d(d^2f) = d\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right) =$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2\right)dy =$$

$$\left(\frac{\partial^3 f}{\partial x^3}dx^2 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dxdy + \frac{\partial^3 f}{\partial x \partial y^2}dy^2\right)dx + \left(\frac{\partial^3 f}{\partial x^2 \partial y}dx^2 + 2\frac{\partial^3 f}{\partial x \partial y^2}dxdy + \frac{\partial^3 f}{\partial y^3}dy^2\right)dy =$$

$$\frac{\partial^3 f}{\partial x^3}dx^3 + 2\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + \frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial x^2 \partial y}dydx^2 + 2\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3 =$$

$$\frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$$

Por lo tanto

$$d^3f = d(d^2f) = \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$$

Diferencial de orden 4

Si $d^3f = \frac{\partial^3 f}{\partial x^3}dx^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 f}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 f}{\partial y^3}dy^3$ entonces una diferencial de orden 4 seria:

$$d^4f = d(d^3f) = \frac{\partial^4 f}{\partial x^4}dx^4 + 4\frac{\partial^4 f}{\partial x^3 \partial y}dx^3dy + 6\frac{\partial^4 f}{\partial x^2 \partial y^2}dx^2dy^2 + 4\frac{\partial^4 f}{\partial x \partial y^3}dxdy^3 + \frac{\partial^4 f}{\partial y^4}dy^4$$

Diferencial de orden n

Vamos a probar por inducción que

$$d^nf = \frac{\partial^n f}{\partial x^n}dx^n + \binom{n}{1}\frac{\partial^{n-1} f}{\partial x^{n-1} \partial y}dx^{n-1}dy + \binom{n}{2}\frac{\partial^{n-2} f}{\partial x^{n-2} \partial y^2}dx^{n-2}dy^2 + \cdots + \binom{n}{k}\frac{\partial^{n-k} f}{\partial x^{n-k} \partial y^k}dx^{n-k}dy^k + \cdots + \frac{\partial^n f}{\partial y^n}dy^n$$

que se puede escribir

$$d^nf = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j} \partial y^j}dx^{n-j}dy^j$$

Demostración. Para n=1 se tiene

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Suponemos valido para n

$$d^nf = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j} \partial y^j}dx^{n-j}dy^j$$

Por demostrar que es valida para n+1

$$d^{n+1}f = d(d^n f) = \frac{\partial}{\partial x}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j\right)dx + \frac{\partial}{\partial y}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j\right)dy =$$

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n-j}\partial y^{j+1}}dx^{n-j}dy^{j+1} =$$

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=1}^{n+1}\binom{n}{j-1}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=1}^{n}\binom{n}{j-1}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\left(\binom{n}{j} + \binom{n}{j-1}\right)\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\binom{n+1}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} = \sum_{j=0}^{n+1}\binom{n}{j}\frac{\partial^{n} f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j$$

□

La última fórmula puede expresarse simbólicamente por la ecuación

$$d^n f = \left(\frac{\partial}{\partial x}\ dx + \frac{\partial}{\partial y}\ dy\right)^n f$$

donde primero debe desarrollarse le expresión de la derecha formalmente por medio del teorema del binomio y, a continuación deben sustituirse los términos

$$\frac{\partial^n f}{\partial x^n}dx^n, \frac{\partial^n f}{\partial x^{n-1}\partial y}dx^{n-1}dy, \cdots, \frac{\partial^n f}{\partial y^n}dy^n$$

por los términos

$$\left(\frac{\partial}{\partial x}dx\right)^n f, \left(\frac{\partial}{\partial x}dx\right)^{n-1}\left(\frac{\partial}{\partial y}dy\right) f, \cdots, \left(\frac{\partial}{\partial y}dy\right)^n f$$

Ejemplo Hallar la diferencial de orden 2 para $f(x,y) = e^{x^2+y^y}$

Solución En este caso tenemos la fórmula

$$d^2 f = d(df) = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dxdy + \frac{\partial^2 f}{\partial y^2}dy^2$$

vamos a calcular las derivadas parciales correspondientes

$$\frac{\partial(e^{x^2+y^2})}{\partial x} = 2xe^{x^2+y^2}$$

$$\frac{\partial(e^{x^2+y^2})}{\partial y} = 2ye^{x^2+y^2}$$

$$\frac{\partial^2(e^{x^2+y^2})}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial(e^{x^2+y^2})}{\partial x}\right) = \frac{\partial(2xe^{x^2+y^2})}{\partial x} = 4x^2e^{x^2+y^2} + 2e^{x^2+y^2}$$

$$\frac{\partial^2(e^{x^2+y^2})}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial(e^{x^2+y^2})}{\partial y}\right) = \frac{\partial(2ye^{x^2+y^2})}{\partial y} = 4y^2e^{x^2+y^2} + 2e^{x^2+y^2}$$

$$\frac{\partial^2(e^{x^2+y^2})}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial(e^{x^2+y^2})}{\partial x}\right) = \frac{\partial(2xe^{x^2+y^2})}{\partial y} = 4xye^{x^2+y^2}$$

$$\frac{\partial^2(e^{x^2+y^2})}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial(e^{x^2+y^2})}{\partial y}\right) = \frac{\partial(2ye^{x^2+y^2})}{\partial x} = 4xye^{x^2+y^2}$$

y la diferencial de orden 2 sería:

$$d^2f = \left(4x^2e^{x^2+y^2} + 2e^{x^2+y^2}\right)dx^2 + 8xye^{x^2+y^2}dxdy + \left(4y^2e^{x^2+y^2} + 2e^{x^2+y^2}\right)dy^2$$