Teorema de Taylor para funciones $f : A \subset \mathbb{R}^2 \to \mathbb{R}$

Sea $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ y sea $F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$ con $t \in [0, 1]$, de esta manera f recorre el segmento de $[x_0, y_0]$ a $[x_0 + h_1 t, y_0 + h_2 t]$. Se tiene entonces que usando la regla de la cadena

$$F'(t) = \frac{\partial f}{\partial x}(x_0 + h_1 t, y_0 + h_2 t) \cdot \frac{d(x_0 + h_1 t)}{dt} + \frac{\partial f}{\partial y}(x_0 + h_1 t, y_0 + h_2 t) \cdot \frac{d(y_0 + h_2 t)}{dt} =$$

$$\frac{\partial f}{\partial x}(x_0 + h_1 t, y_0 + h_2 t) \cdot h_1 + \frac{\partial f}{\partial y}(x_0 + h_1 t, y_0 + h_2 t) \cdot h_2$$

Ejemplo Para $F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$, hallar $F'(1)$ para $f(x, y) = \operatorname{sen}(x + y)$

Solución En este caso tenemos que

$$F(t) = f(x_0 + h_1 t, y_0 + h_2 t) = \operatorname{sen}(x_0 + h_1 t, y_0 + h_2 t) \ \to \ F'(t) =$$

$$\cos(x_0 + h_1 t, y_0 + h_2 t) \cdot h_1 + \cos(x_0 + h_1 t, y_0 + h_2 t) \cdot h_2 = (h_1 + h_2) \cos(x_0 + h_1 t, y_0 + h_2 t)$$

y por lo tanto

$$F'(1) = (h_1 + h_2) \cos(x_0 + h_1, y_0 + h_2)$$

Ejemplo Encontrar la pendiente de la curva $z(t) = F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$, en $t = 1$ para $f(x, y) = x^2 + y^2$, $x = 0$, $y = 1$, $h_1 = \frac{1}{2}$ y $h_2 = \frac{1}{4}$

Solución En este caso tenemos que

$$F(t) = f(x_0 + h_1 t, y_0 + h_2 t) = (x_0 + h_1 t)^2 + (y_0 + h_2 t)^2$$

por lo tanto

$$F'(t) = 2(x_0 + h_1 t)^2 h_1 + (y_0 + h_2 t)^2 \cdot h_2$$

sustituimos valores y obtenemos

$$F'(1) = \frac{1}{2} + \frac{5}{8} = \frac{9}{8}$$

Vamos ahora a calcular $F''(t)$

$$F''(t) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2\right)h_1 + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}h_1 + \frac{\partial f}{\partial y}h_2\right)h_2 =$$

$$\frac{\partial^2 f}{\partial x^2}h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}h_1 h_2 + \frac{\partial^2 f}{\partial y^2}h_2^2$$

simbólicamente se puede escribir

$$F''(t) = \left(\frac{\partial}{\partial x} \cdot h_1 + \frac{\partial}{\partial y} \cdot h_2\right)^2 f$$

y en general

$$F^n(t) = \frac{\partial^n f}{\partial x^n} h_1^n + \binom{n}{1} \frac{\partial^{n-1} f}{\partial x^{n-1} \partial y} h_1^{n-1} h_2 + \binom{n}{2} \frac{\partial^{n-2} f}{\partial x^{n-2} \partial y^2} h_1^{n-2} h_2^2 + \cdots + \binom{n}{k} \frac{\partial^{n-k} f}{\partial x^{n-k} \partial y^k} h_1^{n-k} h_2^k + \cdots + \frac{\partial^n f}{\partial y^n} h_2^n$$

que simbólicamente se puede escribir

$$F^n = \sum_{j=0}^{n} \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial y^j} h_1^{n-j} h_2^j = \left(\frac{\partial}{\partial x} \cdot h_1 + \frac{\partial}{\partial y} \cdot h_2 \right)^n f$$

Ahora bien si se aplica la fórmula de Taylor con la forma del residuo de Lagrange a la función

$$F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$$

y ponemos $t = 1$, $x = x_0 + h_1$, $y_0 + h_2 = y$ se tiene la fórmula de Tatlor para dos variables

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0) \cdot h_1 + \frac{\partial f}{\partial y}(x_0,y_0) \cdot h_2 \right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0) h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(x_0,y_0) h_2^2 \right)$$

$$+ \cdots + \frac{1}{n!}\left(\sum_{j=0}^{n+1} \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial y^j}(x_0,y_0) h_1^{n-j} h_2^j \right)$$

si $h_1 = x - x_0$ y $h_2 = y - y_0$ entonces

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0) \right) +$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y - y_0)^2 \right) +$$

$$\cdots + \frac{1}{n!}\left(\sum_{j=0}^{n+1} \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial y^j}(x_0,y_0)(x - x_0)^{n-j}(y - y_0)^j \right) + R_n$$

donde

$$R_n = \frac{1}{n+1!}\left((x - x_0)^{n+1} \frac{\partial^{n+1} f}{\partial x^{n+1}}(\xi,\eta) + \cdots + (y - y_0)^{n+1} \frac{\partial^{n+1} f}{\partial y^{n+1}}(\xi,\eta) \right)$$

donde $\xi \in (x_0, x_0 + h_1)$ y $\eta \in (y_0, y_0 + h_2)$
En general el residuo R_n se anula en un orden mayor que el término $d^n f$

Ejemplo Obtener el polinomio de Taylorde tercer orden para la función $f(x,y) = \operatorname{sen}(x+y) + \cos(x+y)$ en $(0,0)$

Solución En este caso el polinomio de Taylor de segundo orden es

$$f(x,y) = f(x_0,y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0) \cdot (x - x_0) + \frac{\partial f}{\partial y}(x_0,y_0) \cdot (y - y_0) \right) +$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y - y_0)^2 \right) +$$

$$\frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}(x-x_0)^3+3\frac{\partial^3 f}{\partial x^2\partial y}(x-x_0)^2(y-y_0)+3\frac{\partial^3 f}{\partial x\partial y^2}(x-x_0)(y-y_0)^2+\frac{\partial^3 f}{\partial y^3}(y-y_0)^3\right)+R_3$$

Vamos a calcular las derivadas parciales

$$\frac{\partial f}{\partial x}\Big|_{(0,0)}=\cos(x+y)-\operatorname{sen}(x+y)\Big|_{(0,0)}=1$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)}=\cos(x+y)-\operatorname{sen}(x+y)\Big|_{(0,0)}=1$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)}=-\operatorname{sen}(x+y)-\cos(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial y^2}\Big|_{(0,0)}=-\operatorname{sen}(x+y)-\cos(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial y\partial x}\Big|_{(0,0)}=-\operatorname{sen}(x+y)-\cos(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^3 f}{\partial x^3}\Big|_{(0,0)}=-\cos(x+y)+\operatorname{sen}(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^3 f}{\partial y^3}\Big|_{(0,0)}=-\cos(x+y)-\operatorname{sen}(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^3 f}{\partial y^2\partial x}\Big|_{(0,0)}=-\cos(x+y)+\operatorname{sen}(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial x\partial y}\Big|_{(0,0)}=-\operatorname{sen}(x+y)-\cos(x+y)\Big|_{(0,0)}=-1$$

$$\frac{\partial^3 f}{\partial x^2\partial y}\Big|_{(0,0)}=-\cos(x+y)+\operatorname{sen}(x+y)\Big|_{(0,0)}=-1$$

$$f(0,0)=1$$

$$\therefore\quad f(x,y)=1+x+y-\frac{x^2}{2}-xy-\frac{y^2}{2}-\frac{x^3}{6}-\frac{x^2y}{2}-\frac{xy^2}{2}-\frac{y^3}{6}$$

Ejemplo Obtener un valor aproximado de $\sqrt{1{,}03}\sqrt[3]{0{,}98}$ utilizando la fórmula de Taylor hasta segundo orden

Solución En esta caso consideramos $f(x,y)=\sqrt{x}\sqrt[3]{y}$ en el punto $(1,1)$ y usando la fórmula de Taylor de segundo orden

$$f(x,y)=f(x_0,y_0)+\frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\right)+$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)(x-x_0)^2+2\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)(x-x_0)(y-y_0)+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)(y-y_0)^2\right)+R_2$$

vamos a calcular las derivadas parciales

$$\frac{\partial f}{\partial x}\Big|_{(1,1)} = \frac{1}{2\sqrt{x}}\sqrt[3]{y}\Big|_{(1,1)} = \frac{1}{2}$$

$$\frac{\partial f}{\partial y}\Big|_{(1,1)} = \sqrt{x}\frac{1}{3}y^{-\frac{2}{3}}\Big|_{(1,1)} = \frac{1}{3}$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(1,1)} = \frac{1}{2}x^{-\frac{3}{2}}\left(-\frac{1}{2}\right)\sqrt[3]{y}\Big|_{(1,1)} = -\frac{1}{4}$$

$$\frac{\partial^2 f}{\partial y^2}\Big|_{(1,1)} = \sqrt{x}\frac{1}{3}\left(-\frac{2}{3}y^{-\frac{5}{3}}\right)\Big|_{(1,1)} = -\frac{2}{9}$$

$$\frac{\partial^2 f}{\partial y \partial x}\Big|_{(1,1)} = \frac{1}{2\sqrt{x}}\frac{1}{3}y^{-\frac{2}{3}}\Big|_{(1,1)} = \frac{1}{6}$$

Por lo tanto

$$f(x,y) = 1 + \frac{1}{2}(x-1) + \frac{1}{3}(y-1) - \frac{1}{8}(x-1)^2 + \frac{1}{6}(x-1)(y-1) - \frac{1}{9}(y-1)^2$$

en consecuencia

$$\sqrt{1{,}03}\,\sqrt[3]{0{,}98} = f(1{,}03, 0{,}98) \approx 1+\frac{1}{2}(0{,}3)+\frac{1}{3}(-0{,}02)-\frac{1}{8}(0{,}0009)+\frac{1}{6}(0{,}0006)-\frac{1}{9}(0{,}0004) \approx 1{,}0080763889$$

Ejemplo Calcular el límite

$$\lim_{(x,y)\to(0,0)} \frac{x\,\text{sen}(y) + y\,\text{sen}(x)}{xy}$$

Usando el polinomio de Taylor de orden 2 para la función $f(x,y) = x\,\text{sen}(y) + y\,\text{sen}(x)$

Solución En este caso

$$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \text{sen}(y) + y\cos(x)\Big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}\Big|_{(0,0)} = x\cos(y) + \text{sen}(x)\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial x^2}\Big|_{(0,0)} = -y\,\text{sen}(x)\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y^2}\Big|_{(1,1)} = -x\,\text{sen}(y)\Big|_{(0,0)} = 0$$

$$\frac{\partial^2 f}{\partial y \partial x}\Big|_{(1,1)} = \cos(y) + \cos()x = \Big|_{(0,0)} = 2$$

por lo tanto $f(x,y) = 2xy$ y entonces

$$\lim_{(x,y)\to(0,0)} \frac{x\,\text{sen}(y) + y\,\text{sen}(x)}{xy} = \lim_{(x,y)\to(0,0)} \frac{2xy}{xy} = 2$$