Extremos Locales parte 2

Para el caso de funciones $f:\mathbb{R}^3\to\mathbb{R}$ tenemos que recordando un poco de la expresión de taylor

$$f(x,y)=f(x_0,y_0)+\left(\frac{\partial f}{\partial x}\right)_p(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_p(y-y_0)+\left(\frac{\partial f}{\partial z}\right)_p(z-z_0)+$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}_p(x-x_0)^2+2\frac{\partial^2 f}{\partial x\partial y}_p(x-x_0)(y-y_0)+\frac{\partial^2 f}{\partial y^2}_p(y-y_0)^2+2\frac{\partial^2 f}{\partial x\partial z}_p(z-z_0)(x-x_0)+2\frac{\partial^2 f}{\partial y\partial z}_p(z-z_0)(y-y_0)\right)$$

$$+\frac{\partial^2 f}{\partial z^2}_p(z-z_0)$$

Haciendo $x-x_0=h_1,\ \ y-y_0=h_2,\ \ z-z_0=h_3$ podemos escribir el término rojo de la siguiente manera

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}h_1^2+2\frac{\partial^2 f}{\partial x\partial y}h_1h_2+\frac{\partial^2 f}{\partial y^2}h_2^2+2\frac{\partial^2 f}{\partial x\partial z}h_3h_1+2\frac{\partial^2 f}{\partial y\partial z}h_3h_2+\frac{\partial^2 f}{\partial z^2}h_3^2\right)$$

y también se puede ver como producto de matrices

$$\frac{1}{2!}(h_1\ h_2\ h_3)\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial z\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\partial y}\\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2}\end{pmatrix}_p\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}$$

Si (x_0,y_0,z_0) es un punto critico de la función entonces en la expresión de Taylor

$$f(x,y)=f(x_0,y_0)+\left(\frac{\partial f}{\partial x}\right)_p(x-x_0)+\left(\frac{\partial f}{\partial y}\right)_p(y-y_0)+\left(\frac{\partial f}{\partial z}\right)_p(z-z_0)$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}_p(x-x_0)^2+2\frac{\partial^2 f}{\partial x\partial y}_p(x-x_0)(y-y_0)+\frac{\partial^2 f}{\partial y^2}_p(y-y_0)^2+2\frac{\partial^2 f}{\partial x\partial z}_p(z-z_0)(x-x_0)+2\frac{\partial^2 f}{\partial y\partial z}_p(z-z_0)(y-y_0)\right)$$

$$+\frac{\partial^2 f}{\partial z^2}_p(z-z_0)(x-x_0)$$

El término

$$\frac{\partial f}{\partial x}_p(x-x_0)+\frac{\partial f}{\partial y}_p(y-y_0)+\frac{\partial f}{\partial z}_p(z-z_0)=0$$

y por lo tanto

$$f(x,y)-f(x_0,y_0)=\frac{1}{2!}(h_1\ h_2\ h_3)\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial z\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\partial y}\\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2}\end{pmatrix}_p\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}$$

vamos a determinar el signo de la forma

$$Q(h)=\frac{1}{2!}(h_1\ h_2\ h_3)\begin{pmatrix}\frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial z\partial x}\\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\partial y}\\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2}\end{pmatrix}_p\begin{pmatrix}h_1\\h_2\\h_3\end{pmatrix}$$

vamos a trabajar sin el término $\dfrac{1}{2!}$ que no afectara al signo de la expresión, tenemos entonces

$$Q(h) = (h_1\ h_2\ h_3)\begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y\partial x} & \frac{\partial^2 f}{\partial z\partial x} \\ \frac{\partial^2 f}{\partial x\partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z\partial y} \\ \frac{\partial^2 f}{\partial x\partial z} & \frac{\partial^2 f}{\partial y\partial z} & \frac{\partial^2 f}{\partial z^2} \end{pmatrix}_p \begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \frac{\partial^2 f}{\partial x^2}h_1^2 + 2\frac{\partial^2 f}{\partial x\partial y}h_1h_2 + \frac{\partial^2 f}{\partial y^2}h_2^2 + 2\frac{\partial^2 f}{\partial x\partial z}h_3h_1 + 2\frac{\partial^2 f}{\partial y\partial z}h_3h_2 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

$$= \frac{\partial^2 f}{\partial x^2}\left(h_1 + \frac{\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}h_2\right)^2 + \left(\frac{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y\partial x}\right)^2}{\frac{\partial^2 f}{\partial x^2}}\right)h_2^2 + 2\frac{\partial^2 f}{\partial x\partial z}h_3h_1 + 2\frac{\partial^2 f}{\partial y\partial z}h_3h_2 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

hacemos $b_1 = \dfrac{\partial^2 f}{\partial x^2}$, $\quad h_1' = \left(h_1 + \dfrac{\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}h_2\right)$, $\quad b_2 = \dfrac{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y\partial x}\right)^2}{\frac{\partial^2 f}{\partial x^2}}$, $\quad h_2' = h_2$ y obtenemos

$$= b_1h_1'^2 + b_2h_2'^2 + 2\frac{\partial^2 f}{\partial x\partial z}h_3h_1 + 2\frac{\partial^2 f}{\partial y\partial z}h_3h_2 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

que podemos escribir

$$= b_1h_1'^2 + b_2h_2'^2 + 2\frac{\partial^2 f}{\partial x\partial z}\left(h_1 + \frac{\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}h_2 - \frac{\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}h_2\right)h_3 + 2\frac{\partial^2 f}{\partial y\partial z}h_3h_2 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

$$= b_1h_1'^2 + b_2h_2'^2 + 2\frac{\partial^2 f}{\partial x\partial z}\left(h_1' - \frac{\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}h_2'\right)h_3 + 2\frac{\partial^2 f}{\partial y\partial z}h_3h_2 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

$$= b_1h_1'^2 + b_2h_2'^2 + 2\frac{\partial^2 f}{\partial x\partial z}h_1'h_3 + \left(2\frac{\partial^2 f}{\partial y\partial z} - \frac{2\frac{\partial^2 f}{\partial x\partial z}\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}\right)h_2'h_3 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

hacemos

$$2b_{23} = 2\frac{\partial^2 f}{\partial y\partial z} - \frac{2\frac{\partial^2 f}{\partial x\partial z}\frac{\partial^2 f}{\partial y\partial x}}{\frac{\partial^2 f}{\partial x^2}}$$

y obtenemos

$$= b_1h_1'^2 + b_2h_2'^2 + 2\frac{\partial^2 f}{\partial x\partial z}h_1'h_3 + 2b_{23}h_2'h_3 + \frac{\partial^2 f}{\partial z^2}h_3^2$$

que se puede escribir

$$= b_1\left(h_1'^2 + 2\frac{\frac{\partial^2 f}{\partial x\partial z}}{b_1}h_1'h_3 + \left(\frac{\frac{\partial^2 f}{\partial x\partial z}h_3}{b_1}\right)^2\right) + b_2\left(h_2'^2 + 2\frac{b_{23}}{b_2}h_2'h_3 + \left(\frac{b_{23}}{b_2}h_3\right)^2\right) + \left(\frac{\partial^2 f}{\partial z^2} - \frac{\left(\frac{\partial^2 f}{\partial x\partial z}\right)^2}{b_1} - \frac{b_{23}^2}{b_2}\right)h_3^2$$

hacemos

$$b_3 = \frac{\partial^2 f}{\partial z^2} - \frac{\left(\frac{\partial^2 f}{\partial x\partial z}\right)^2}{b_1} - \frac{b_{23}^2}{b_2}$$

y obtenemos

$$= b_1 \left(h_1'^2 + 2\frac{\frac{\partial^2 f}{\partial x \partial z}}{b_1} h_1' h_3 + \left(\frac{\frac{\partial^2 f}{\partial x \partial z} h_3}{b_1} \right)^2 \right) + b_2 \left(h_2'^2 + 2\frac{b_{23}}{b_2} h_2' h_3 + \left(\frac{b_{23}}{b_2} h_3 \right)^2 \right) + b_3 h_3^2$$

$$= b_1 \left(h_1' + \frac{\frac{\partial^2 f}{\partial x \partial z}}{b_1} h_3 \right)^2 + b_2 \left(h_2' + \frac{b_{23}}{b_2} h_3 \right)^2 + b_3 h_3^2$$

esta última expresión será positiva si y solo si $b_1 > 0$ $b_2 > 0$ y $b_3 > 0$ en clases pasadas vimos los dos primeros, veamos ahora que

$$b_3 = \frac{\partial^2 f}{\partial z^2} - \frac{\left(\frac{\partial^2 f}{\partial x \partial z} \right)^2}{b_1} - \frac{b_{23}^2}{b_2} > 0$$

tenemos entonces que

$$\frac{\partial^2 f}{\partial z^2} - \frac{\left(\frac{\partial^2 f}{\partial x \partial z} \right)^2}{b_1} - \frac{b_{23}^2}{b_2} = \frac{\partial^2 f}{\partial z^2} - \frac{\left(\frac{\partial^2 f}{\partial x \partial z} \right)^2}{\frac{\partial^2 f}{\partial z^2}} - \frac{\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{2\frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x}}{\frac{\partial^2 f}{\partial x^2}} \right)^2}{\frac{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\frac{\partial^2 f}{\partial x^2}}}$$

$$= \frac{\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial x^2} \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2}{\frac{\partial^2 f}{\partial x^2}} - \frac{\frac{\left(\frac{\partial^2 f}{\partial y \partial z}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\left(\frac{\partial^2 f}{\partial x^2} \right)^2}}{\frac{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\frac{\partial^2 f}{\partial x^2}}} = \frac{\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial x^2} \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2}{\frac{\partial^2 f}{\partial x^2}} - \frac{\left(\frac{\partial^2 f}{\partial y \partial z}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 \right)\frac{\partial^2 f}{\partial x^2}}$$

$$= \frac{\left(\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2 \right)\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 \right) - \left(\frac{\partial^2 f}{\partial y \partial z}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 \right)}$$

$$= \frac{\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 - \frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial x \partial z} \right)^2 + \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2\left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 - \left(\frac{\partial^2 f}{\partial y \partial z} \right)^2\left(\frac{\partial^2 f}{\partial x^2} \right)^2 -}{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 \right)}$$

$$\frac{2\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y \partial z}\frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x} \right) - \left(\frac{\partial^2 f}{\partial x \partial z} \right)^2\left(\frac{\partial^2 f}{\partial y \partial x} \right)^2}{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 \right)}$$

$$= \frac{\frac{\partial^2 f}{\partial z^2}\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial z^2}\left(\frac{\partial^2 f}{\partial y \partial x} \right)^2 - \frac{\partial^2 f}{\partial y^2}\left(\frac{\partial^2 f}{\partial x \partial z} \right)^2 - \left(\frac{\partial^2 f}{\partial y \partial z} \right)^2\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial y \partial z}\frac{\partial^2 f}{\partial x \partial z}\frac{\partial^2 f}{\partial y \partial x}}{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x} \right)^2}$$

$$= \frac{\begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix}}{\frac{\partial^2 f}{\partial y^2}\frac{\partial^2 f}{\partial x^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2}$$

por lo tanto

$$b_3 > 0 \;\Leftrightarrow\; \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial z \partial x} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} & \frac{\partial^2 f}{\partial z \partial y} \\ \frac{\partial^2 f}{\partial x \partial z} & \frac{\partial^2 f}{\partial y \partial z} & \frac{\partial^2 f}{\partial z^2} \end{vmatrix} > 0$$

Definición 1. *La forma $Q(x) = xAx^t$, que tiene asociada la matriz A (respecto a la base canónica de $\mathbb{R}^n$) se dice:*
Definida positiva, si $Q(x) > 0 \quad \forall \quad x \in \mathbb{R}^n$
La forma $Q(x) = xAx^t$, que tiene asociada la matriz A (respecto a la base canónica de $\mathbb{R}^n$) se dice:
Definida negativa, si $Q(x) < 0 \quad \forall \quad x \in \mathbb{R}^n$

Definición 2. *Si la forma $Q(x) = xAx^t$ es definida positiva, entonces f tiene un mínimo local en en x*
Si la forma $Q(x) = xAx^t$ es definida negativa, entonces f tiene un máximo local en en x

Hay criterios similares para una matriz simetrica A de $n \times n$ y consideramos las n submatrices cuadradas a lo largo de la diagonal, A es definida positiva si y solo si los determinantes de estas submatrices diagonales son todos mayores que cero. Para A definida negativa los signos deberan alternarse < 0 y > 0. En casi de que los determinantes de las submatrices diagonales sean todos diferentes de cero pero que la matrix no sea definida positiva o negativa, el punto crítico es tipo silla. Y por lo tanto el punto no es máximo ni mínimo. Asi tenemos el siguiente resultado.

$$\begin{bmatrix}\begin{bmatrix}\begin{bmatrix}\begin{bmatrix}\ \end{bmatrix} & \\ & \end{bmatrix} & \\ & \end{bmatrix} & \\ & \end{bmatrix}$$

Definición 3. *Dada una matriz cuadrada $A = a_{ij} \quad j = 1, ..., n \quad i = 1, ..., n$ se consideran las submatrices angulares $A_k \quad k = 1, ..., n$ definidas como*

$$A_1 = (a_{11}) \quad A_2 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad A_3 = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \cdots, A_n = A$$

se define $\det A_k = \triangle_k$

Definición 4. *Se tiene entonces que que la forma $Q(x) = xAX^t$ es definida positiva si y solo si todos los dterminantes $\triangle_k \quad k = 1, ..., n$ son números positivos*

Definición 5. *La forma $Q(x) = xAX^t$ es definida negativa si y solo si los dterminantes $\triangle_k \quad k = 1, ..., n$ tienen signos alternados comenzando por $\triangle_1 < 0, \quad \triangle_2 > 0, ...$ respectivamente*

Ejemplo: Consideremos la función $f : \mathbb{R}^3 \to \mathbb{R}$ $f(x, y, z) = \sin x + \sin y + \sin z - \sin(x + y + z)$, el punto $P = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ es un punto crítico de f y en ese punto la matriz hessiana de f es

$$H(p) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

los determinantes de las submatrices angulares son

$$\Delta_1 = det(-2)$$

$$\Delta_2 = det \begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}$$

$$\Delta_3 = det H(p) = -4$$

puesto que son signos alternantes con $\Delta t < 0$ concluimos que la funcion f tiene en $\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right)$ un máximo local. Este máximo local vale $f\left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right) = 4$