08/04/2014

Corrección del problema 1) c) del día 03/04/2014:

$$D_{(\pi,\frac{\pi}{2})} f(x,y) \quad \text{con } f(x,y) = y \operatorname{Sen} x$$

$$\Rightarrow D_{(\pi,\frac{\pi}{2})} f(x,y) = \lim_{t\to 0} \frac{f((x,y)+t(\pi,\frac{\pi}{2})) - f(x,y)}{t} = \lim_{t\to 0} \frac{f(x+\pi t, y+\frac{\pi}{2}t) - f(x,y)}{t}$$

$$= \lim_{t\to 0} \frac{(y+\frac{\pi}{2}t)\operatorname{Sen}(x+\pi t) - y\operatorname{Sen}x}{t}$$

$$= \lim_{t\to 0} \frac{y(\operatorname{Sen}x\operatorname{Cos}\pi t + \operatorname{Sen}\pi t \operatorname{Cos}x) + \frac{\pi}{2}t(\operatorname{Sen}x\operatorname{Cos}\pi t + \operatorname{Sen}\pi t\operatorname{Cos}x) - y\operatorname{Sen}x}{t}$$

$$= \lim_{t\to 0} \left(\frac{y\operatorname{Sen}x\operatorname{Cos}\pi t - y\operatorname{Sen}x}{t} + \underbrace{\frac{\pi y\operatorname{Sen}\pi t\operatorname{Cos}x}{\pi t}}_{\to\ \pi y\operatorname{Cos}x} + \underbrace{\frac{\pi}{2}\operatorname{Sen}x\operatorname{Cos}\pi t}_{\to\ \frac{\pi}{2}\operatorname{Sen}x} + \underbrace{\frac{\pi}{2}\operatorname{Sen}\pi t\operatorname{Cos}x}_{\to\ 0} \right.$$

$$= \lim_{t\to 0} \underbrace{-\pi y\operatorname{Sen}x\operatorname{Sen}\pi t}_{\to\ 0} + \pi y\operatorname{Cos}x + \frac{\pi}{2}\operatorname{Sen}x$$

↓
aplicando L'Hôpital al 1er. sumando.

$$\therefore\ D_{(\pi,\frac{\pi}{2})} y\operatorname{Sen}x = \pi y\operatorname{Cos}x + \frac{\pi}{2}\operatorname{Sen}x$$

Propiedades del ∇

1) $\nabla(f+g) = \nabla f + \nabla g$
2) $\nabla(fg) = f\nabla g + g\nabla f$
3) $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$ con $g \neq 0$

Dem.

1) $\nabla(f+g) = (\partial_x, \partial_y, \partial_z)(f+g) = (\partial_x(f+g), \partial_y(f+g), \partial_z(f+g))$

$= (\partial_x f + \partial_x g, \partial_y f + \partial_y g, \partial_z f + \partial_z g) = (\partial_x f, \partial_y f, \partial_z f) + (\partial_x g, \partial_y g, \partial_z g)$

$= (\partial_x, \partial_y, \partial_z) f + (\partial_x, \partial_y, \partial_z) g = \nabla f + \nabla g$

2) $\nabla(fg) = (\partial_x, \partial_y, \partial_z)(fg) = (\partial_x(fg), \partial_y(fg), \partial_z(fg))$

$= (f\partial_x g + g\partial_x f, f\partial_y g + g\partial_y f, f\partial_z g + g\partial_z f)$

$= (f\partial_x g, f\partial_y g, f\partial_z g) + (g\partial_x f, g\partial_y f, g\partial_z f)$

$= f(\partial_x, \partial_y, \partial_z)g + g(\partial_x, \partial_y, \partial_z)f = \underline{f\nabla g + g\nabla f}$

3) $\nabla\left(\frac{f}{g}\right) = (\partial_x, \partial_y, \partial_z)\left(\frac{f}{g}\right) = \left(\partial_x\left(\frac{f}{g}\right), \partial_y\left(\frac{f}{g}\right), \partial_z\left(\frac{f}{g}\right)\right)$

$= \left(\frac{g\partial_x f - f\partial_x g}{g^2}, \frac{g\partial_y f - f\partial_y g}{g^2}, \frac{g\partial_z f - f\partial_z g}{g^2}\right)$

$= \left(\frac{g\partial_x f}{g^2}, \frac{g\partial_y f}{g^2}, \frac{g\partial_z f}{g^2}\right) - \left(\frac{f\partial_x g}{g^2}, \frac{f\partial_y g}{g^2}, \frac{f\partial_z g}{g^2}\right)$

$= \frac{g}{g^2}(\partial_x, \partial_y, \partial_z)f - \frac{f}{g^2}(\partial_x, \partial_y, \partial_z)g = \frac{g\nabla f}{g^2} - \frac{f\nabla g}{g^2}$

$= \underline{\frac{g\nabla f - f\nabla g}{g^2}}$

Derivadas direccionales usando ∇.

$$\frac{\partial f}{\partial \bar{u}} = \nabla f \cdot \bar{u}$$

a) $f(x,y,z) = x^2y^3z^4$, $(x_0, y_0, z_0) = (1,1,1)$ y $\bar{u} = \frac{1}{\sqrt{31}}(1,1,1)$

b) $f(x,y) = xy$ $\quad \bar{u} = (1,1)$

c) $f(x,y) = \frac{y}{x}$ $\quad \bar{u} = (1,2)$

d) $f(x,y) = y^2 \ln x$ $\quad \bar{u} = (1,1)$

Sol.

a) $\nabla f(x_0,y_0,z_0) = (2x_0y_0^3z_0^4, 3x_0^2y_0^2z_0^4, 4x_0^2y_0^3z_0^3) = (2,3,4)$

$\Rightarrow \nabla f(x_0,y_0,z_0)\cdot\bar{u} = (2,3,4)\cdot\left(\frac{1}{\sqrt{31}}, \frac{1}{\sqrt{31}}, \frac{1}{\sqrt{31}}\right) = \frac{9}{\sqrt{31}} = \underline{3\sqrt{31}}$

08/04/2014

b) $\nabla f = (y, x) \Rightarrow \nabla f \cdot \bar{u} = (y, x)\cdot(1,1) = x + y$

c) $\nabla f = \left(-\frac{y}{x^2}, \frac{1}{x}\right) \Rightarrow \nabla f \cdot \bar{u} = \left(-\frac{y}{x^2}, \frac{1}{x}\right)\cdot(1,2) = \frac{2}{x} - \frac{y}{x^2}$

d) $\nabla f = \left(\frac{y^2}{x}, 2y\ln x\right) \Rightarrow \nabla f \cdot \bar{u} = \left(\frac{y^2}{x}, 2y\ln x\right)\cdot(1,1) = 2y\ln x + \frac{y^2}{x}$

Regla de la cadena

$c: \mathbb{R} \to \mathbb{R}^3$ dada por $c(t) = (x(t), y(t), z(t))$

$f: \mathbb{R}^3 \to \mathbb{R}$ y $h: \mathbb{R} \to \mathbb{R}$ dada por $h(t) = f\circ c(t) = f(x(t), y(t), z(t))$

$\Rightarrow \partial_t h = \partial_x f\, \partial_t x + \partial_y f\, \partial_t y + \partial_z f\, \partial_t z$

a) $f(x,y,z) = xy^2 \ln z$ con $x(t) = 6t$, $y(t) = t^2$, $z(t) = e^{e^t}$

$\partial_x f = y^2 \ln z = t^4 e^t$ $\quad \partial_y f = 2xy\ln z = 12t^3 e^t$ $\quad \partial_z f = \frac{xy^2}{z} = 6t^5 e^{-e^t}$

$\partial_t x = 6$ $\quad \partial_t y = 2t$ $\quad \partial_t z = e^t e^{e^t}$

$\therefore \partial_t h = 6t^4 e^t + 24t^4 e^t + 6t^5 e^t = 6t^4 e^t(5 + t)$

b) $f(x,y,z) = y^x \log_{10} z$ con $x(t) = 2^t$, $y(t) = e^t$, $z(t) = 10^{10^t}$

$\partial_x f = \partial_x\left(e^{x\ln y}\log_{10} z\right) = y^x \ln y \log_{10} z = t\, e^{2^t t}\, 10^t$

$\partial_t x = \partial_t\left(e^{t\ln 2}\right) = 2^t \ln 2$

$\partial_y f = x y^{x-1} \log_{10} z = 2^t e^{2^t t - t}\, 10^t$

$\partial_t y = e^t$

$\partial_z f = \partial_z\left(y^x \log_{10} e^{\ln z}\right) = \partial_z\left(y^x \ln z \log_{10} e\right) = \frac{y^x \log_{10} e}{z} = e^{2^t t} \log_{10} e\; 10^{-10^t}$

$\partial_t z = \partial_t\left(e^{10^t \ln 10}\right) = \partial_t\left(e^{e^{t\ln 10}\ln 10}\right) = 10^t\, 10^{10^t} (\ln 10)^2$

$\therefore \partial_t h = t e^{2^t t}\, 10^t\, 2^t \ln 2 + 2^t e^{2^t t}\, 10^t + 10^t e^{2^t t} (\ln 10)^2 \log_{10} e$

$= 10^t e^{2^t t}\left(t 2^t \ln 2 + 2^t + (\ln 10)^2 \log_{10} e\right)$