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Taylor

$$f(\bar{x}+\bar{a}) = f(\bar{a}) + \sum_{p=1}^{m} \frac{1}{p!}\left[(x_1-a_1)\partial_{x_1} + (x_2-a_2)\partial_{x_2} + \dots + (x_n-a_n)\partial_{x_n}\right]^p f(\bar{a})$$

$$+ R_{m+1}(\bar{a})$$

donde $R_{m+1}(\bar{a}) = \frac{1}{(m+1)!}\left[(x_1-a_1)\partial x_1 + \dots + (x_n-a_n)\partial x_n\right]^{m+1} f(\bar{a})$

para $f: \mathbb{R}^n \to \mathbb{R}$

a) $f(x) = \operatorname{sen} x$ Taylor a 5º orden en $a=0$

$\Rightarrow f(a) = 0$

$\partial_x f|_a = \cos x|_a = 1$

$\partial_x^2 f|_a = -\operatorname{sen} x|_a = 0$

$\partial_x^3 f|_a = -\cos x|_a = -1$

$\partial_x^4 f|_a = \operatorname{sen} x|_a = 0$

$\partial_x^5 f|_a = \cos x|_a = 1$

$\therefore \operatorname{sen} x = 0 + x + \cancel{\frac{x^2}{2!}}(0) - \frac{x^3}{3!} + \cancel{\frac{x^4}{4!}}(0) + \frac{x^5}{5!}$

$\therefore \operatorname{sen} x = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

b) $f(x) = \cos x$ Taylor a 4º orden en $a=0$

$\Rightarrow f(a) = 1$

$\partial_x f|_a = -\operatorname{sen} x|_a = 0$

$\partial_x^2 f|_a = -\cos x|_a = -1$

$\partial_x^3 f|_a = \operatorname{sen} x|_a = 0$

$\partial_x^4 f|_a = \cos x|_a = 1$

$\therefore \cos x = 1 + \cancel{x}(0) - \frac{x^2}{2!} + \cancel{\frac{x^3}{3!}}(0) + \frac{x^4}{4!}$

$\therefore \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$

c) $f(x) = e^x$ Taylor a 5º orden en $a=0$

$\Rightarrow f(a) = 1$

$\partial_x f|_a = e^x|_a = 1$

$\partial_x^2 f|_a = e^x|_a = 1$

$\partial_x^3 f|_a = e^x|_a = 1$

$\partial_x^4 f|_a = e^x|_a = 1$

$\partial_x^5 f|_a = e^x|_a = 1$

$\therefore e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$

d) $f(x,y) = \ln x \ln y$ Taylor a 3º orden en $a=(1,1)$

$\Rightarrow f(\bar{a}) = 0$

$$\partial_x f\big|_{\bar a} = \frac{\ln y}{x}\Big|_{\bar a} = 0 \qquad \partial_x^2 f\big|_{\bar a} = -\frac{\ln y}{x^2}\Big|_{\bar a} = 0 \qquad \partial_x^3 f\big|_{\bar a} = \frac{2\ln y}{x^3}\Big|_{\bar a} = 0$$

$$\partial_y f\big|_{\bar a} = \frac{\ln x}{y}\Big|_{\bar a} = 0 \qquad \partial_{xy}^2 f\big|_{\bar a} = \frac{1}{xy}\Big|_{\bar a} = 1 \qquad \partial_{xxy}^3 f\big|_{\bar a} = -\frac{1}{x^2 y}\Big|_{\bar a} = -1$$

$$\partial_y^2 f\big|_{\bar a} = -\frac{\ln x}{y^2}\Big|_{\bar a} = 0 \qquad \partial_{xyy}^3 f\big|_{\bar a} = -\frac{1}{xy^2}\Big|_{\bar a} = -1$$

$$\partial_y^3 f\big|_{\bar a} = \frac{2\ln x}{y^3}\Big|_{\bar a} = 0$$

$$\therefore \ln(x+1)\ln(y+1) = 0 + \cancel{(x-1)(0)} + \cancel{(y-1)(0)} + \frac{1}{2}\left[\cancel{(x-1)^2(0)} + 2(x-1)(y-1) + \cancel{(y-1)^2(0)}\right]$$
$$+ \frac{1}{3!}\left[\cancel{(x-1)^3(0)} + 3(x-1)^2(y-1)(-1) + 3(x-1)(y-1)^2(-1) + \cancel{(y-1)^3(0)}\right]$$

$$\therefore \underline{\ln(x+1)\ln(y+1) = (x-1)(y-1) - \frac{1}{2}\left[(x-1)^2(y-1) + (x-1)(y-1)^2\right]}$$

e) $f(x,y) = e^{x^2} + e^{y^2}$ Taylor a 2º orden en $a=(0,0)$

$\Rightarrow f(\bar a) = 2$

$$\partial_x f\big|_{\bar a} = 2xe^{x^2}\Big|_{\bar a} = 0 \qquad \partial_x^2 f\big|_{\bar a} = (2+4x^2)e^{x^2}\Big|_{\bar a} = 2$$

$$\partial_y f\big|_{\bar a} = 2ye^{y^2}\Big|_{\bar a} = 0 \qquad \partial_{xy}^2 f\big|_{\bar a} = 0$$

$$\partial_y^2 f\big|_{\bar a} = (2+4y^2)e^{y^2}\Big|_{\bar a} = 2$$

$$\therefore e^{x^2} + e^{y^2} = 2 + \cancel{x(0)} + \cancel{y(0)} + \frac{1}{2}\left[x^2(2) + \cancel{2xy(0)} + y^2(2)\right]$$

$$\therefore \underline{e^{x^2} + e^{y^2} = 2 + x^2 + y^2}$$

f) $f(x,y) = \frac{x}{y} + \frac{y}{x}$ Taylor a 2º orden en $a=(\sqrt{2},\sqrt{2})$

$\Rightarrow f(\bar a) = 2$

$$\partial_x f\big|_{\bar a} = \frac{1}{y} - \frac{y}{x^2}\Big|_{\bar a} = \frac{1}{\sqrt{2}} - \frac{\sqrt{2}}{2} = 0 \qquad \partial_x^2 f\big|_{\bar a} = \frac{2y}{x^3}\Big|_{\bar a} = 1 \qquad \partial_y^2 f\big|_{\bar a} = \frac{2x}{y^3}\Big|_{\bar a} = 1$$

$$\partial_y f\big|_{\bar a} = -\frac{x}{y^2} + \frac{1}{x}\Big|_{\bar a} = 0 \qquad \partial_{xy}^2 f\big|_{\bar a} = -\frac{1}{y^2} - \frac{1}{x^2}\Big|_{\bar a} = -1$$

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$$\therefore \frac{x+\sqrt{21}}{y+\sqrt{21}}+\frac{y+\sqrt{21}}{x+\sqrt{21}}=2+\frac{1}{2}\left[(x-\sqrt{21})^2-2(x-\sqrt{21})(y-\sqrt{21})+(y-\sqrt{21})^2\right]$$

g) $f(x,y)=x^3+xy+y^3$ Taylor a 8º orden en $a=(0,0)$

$f(\bar{a})=0$

$\partial_x f|_{\bar{a}}=3x^2+y|_{\bar{a}}=0$ $\quad \partial_x^2 f|_{\bar{a}}=6x|_{\bar{a}}=0$ $\quad \partial_x^3 f|_{\bar{a}}=6$

$\partial_y f|_{\bar{a}}=x+3y^2|_{\bar{a}}=0$ $\quad \partial_{xy}^2 f|_{\bar{a}}=1$ $\quad \partial_{xxy}^3 f|_{\bar{a}}=0$

$\partial_y^2 f|_{\bar{a}}=6y|_{\bar{a}}=0$ $\quad \partial_{xyy}^3 f|_{\bar{a}}=0$

$\partial_y^3 f|_{\bar{a}}=6$

$$\therefore x^3+xy+y^3=0+0+0+\frac{1}{2}[0+2xy+0]+\frac{1}{6}[6x^3+0+0+6y^3]$$

$$\therefore x^3+xy+y^3=xy+x^3+y^3$$

h) Dar un valor aproximado de $(0.98)^8\ln(1.01)$ con Taylor a 3º orden

Proponemos la función $f(x,y)=x^8\ln y$ y hacemos la expansión en Taylor alrededor del entero más cercano a ambos valores, i,e, en torno a $a=(1,1)$.

$\Rightarrow f(\bar{a})=0$

$\partial_x f|_{\bar{a}}=8x^7\ln y|_{\bar{a}}=0$ $\quad \partial_x^2 f|_{\bar{a}}=56x^6\ln y|_{\bar{a}}=0$ $\quad \partial_x^3 f|_{\bar{a}}=336x^5\ln y|_{\bar{a}}=0$

$\partial_y f|_{\bar{a}}=\frac{x^8}{y}\Big|_{\bar{a}}=1$ $\quad \partial_{xy}^2 f|_{\bar{a}}=\frac{8x^7}{y}\Big|_{\bar{a}}=8$ $\quad \partial_{xxy}^3 f|_{a}=56\frac{x^6}{y}\Big|_{\bar{a}}=56$

$\partial_y^2 f|_{\bar{a}}=-\frac{x^8}{y^2}\Big|_{\bar{a}}=-1$ $\quad \partial_{xyy}^3 f|_{\bar{a}}=-\frac{8x^7}{y^2}\Big|_{\bar{a}}=-8$

$\partial_y^3 f|_{\bar{a}}=\frac{2x^8}{y^3}\Big|_{\bar{a}}=2$

$$\therefore (x+1)^8\ln(y+1)=(y-1)+\frac{1}{2}\left[16(x-1)(y-1)-(y-1)^2\right]$$
$$+\frac{1}{6}\left[168(x-1)^2(y-1)-24(x-1)(y-1)^2+2(y-1)^3\right]$$

Ahora, si evaluamos $x=0.98$ y $y=1.01$ en el desarrollo en Taylor resulta que

$$= (0.01) + \frac{1}{2}\left[16(-0.02)(0.01) - (0.01)^2\right] + \frac{1}{6}\left[168(-0.02)^2(0.01) - 24(-0.02)(0.01)^2 + 2(0.01)^3\right]$$

$$= 0.008470333$$

y el valor de $(0.98)^8 \ln(1.01) = 0.008465373$

en el cual se aprecia que el valor estimado con Taylor es muy próximo al valor real.

i) $\lim\limits_{(x,y)\to(0,0)} \dfrac{f(x,y)}{xy}$ con $f(x,y) = y\tan x$ con Taylor a 3º orden.

$a=(0,0)$

$\Rightarrow f(\bar{a}) = 0$

$\partial_x f|_{\bar{a}} = y\sec^2 x|_{\bar{a}} = 0 \qquad \partial_x^2 f|_{\bar{a}} = 2y\sec^2 x\tan x|_{\bar{a}} = 0$

$\partial_y f|_{\bar{a}} = \tan x|_{\bar{a}} = 0 \qquad \partial_{xy}^2 f|_{a} = \sec^2 x|_{\bar{a}} = 1$

$\partial_y^2 f|_{\bar{a}} = 0$

$\partial_x^3 f|_{\bar{a}} = 2y(2\sec^2 x\tan^2 x + \sec^4 x)|_{\bar{a}} = 0$

$\partial_{xxy}^3 f|_{\bar{a}} = 2\sec^2 x\tan x|_{\bar{a}} = 0$

$\partial_{xyy}^3 f|_{\bar{a}} = 0$

$\partial_y^3 f|_{\bar{a}} = 0$

$\therefore\ y\tan x = xy$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{f(x,y)}{xy} = \lim_{(x,y)\to(0,0)} \frac{xy}{xy} = 1$$