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1) Diferenciabilidad

Una función $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ es diferenciable en $\bar{x}_0 \in \Omega$ si

$$\lim_{\bar{h}\to\bar{0}} \frac{\| f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} \|}{\|\bar{h}\|} = 0$$ donde $Jf(\bar{x}_0)$ es el jacobiano de f evaluado en $\bar{x}_0$.

a) $f(x,y) = (xy, x+y, x-y)$ con $x_0 = (0,0)$

b) $f(x,y,z) = (x^2-y^2, x+y+z)$ con $x_0 = (1,1,1)$

Sol.

a) $f(\bar{x}_0+\bar{h}) = f(h_1,h_2) = (h_1h_2, h_1+h_2, h_1-h_2)$

$f(\bar{x}_0) = (0,0,0)$

$$Jf(\bar{x}_0) = \begin{pmatrix} y & x \\ 1 & 1 \\ 1 & -1 \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix} \quad y \quad Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \begin{pmatrix} 0 \\ h_1+h_2 \\ h_1-h_2 \end{pmatrix}$$

$$\therefore f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} h_1h_2 \\ h_1+h_2 \\ h_1-h_2 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ h_1+h_2 \\ h_1-h_2 \end{pmatrix} = \begin{pmatrix} h_1h_2 \\ 0 \\ 0 \end{pmatrix} \cdots (1)$$

$\Rightarrow \| (1) \| = |h_1h_2|$

$$\Rightarrow \lim_{(h_1,h_2)\to(0,0)} \frac{|h_1h_2|}{\sqrt{h_1^2+h_2^2}} = \lim_{r\to0} \frac{r^2|\cos\theta \operatorname{sen}\theta|}{r} = 0$$

cambio a polares

$\forall \varepsilon > 0 \ \exists \delta > 0 \ \ni \|\bar{h}\| < \delta \Rightarrow \left| \frac{h_1h_2}{\|\bar{h}\|} \right| < \varepsilon$

$|h_1| \leq \|\bar{h}\|$

$|h_2| \leq \|\bar{h}\| < \delta$ $\Rightarrow \left| \frac{h_1h_2}{\|\bar{h}\|} \right| \leq \left| \frac{h_2\cancel{\|\bar{h}\|}}{\cancel{\|\bar{h}\|}} \right| = |h_2| < \delta = \varepsilon$ $\therefore$ basta con tomar $\delta = \varepsilon$

$\therefore f(x,y)$ es diferenciable en $(0,0)$ ∎

b) $f(\bar{x}_0+\bar{h}) = f(1+h_1, 1+h_2, 1+h_3) = (2h_1-2h_2+h_1^2-h_2^2,\ 3+h_1+h_2+h_3)$

$f(\bar{x}_0) = (0,3)$

$$Jf(\bar{x}_0) = \begin{pmatrix} 2x & -2y & 0 \\ 1 & 1 & 1 \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix} \quad y \quad Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} 2 & -2 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 2h_1-2h_2 \\ h_1+h_2+h_3 \end{pmatrix}$$

$$\therefore f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} 2h_1-2h_2+h_1^2-h_2^2 \\ 3+h_1+h_2+h_3 \end{pmatrix} - \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 2h_1-2h_2 \\ h_1+h_2+h_3 \end{pmatrix} = \begin{pmatrix} h_1^2-h_2^2 \\ 0 \end{pmatrix}$$

y su norma es $|h_1^2-h_2^2|$

$$\Rightarrow \lim_{(h_1,h_2,h_3)\to(0,0,0)} \frac{|h_1^2-h_2^2|}{\sqrt{h_1^2+h_2^2+h_3^2}} \underset{\text{cambio a esféricas}}{=} \lim_{r\to 0} \frac{r^2 \operatorname{sen}^2\theta\,|\cos^2\varphi - \operatorname{sen}^2\varphi|}{r} = 0$$

$\forall \varepsilon>0\ \exists \delta>0 \ \dagger\ \|\bar{h}\|<\delta \Rightarrow \left|\dfrac{h_1^2-h_2^2}{\|\bar{h}\|}\right| < \varepsilon$

$$\Rightarrow \left|\frac{h_1^2-h_2^2}{\|\bar{h}\|}\right| \leq \left|\frac{h_1^2}{\|\bar{h}\|}\right| + \left|\frac{h_2^2}{\|\bar{h}\|}\right| \leq \left|\frac{\|\bar{h}\|^{\cancel{2}}}{\cancel{\|\bar{h}\|}}\right| + \left|\frac{\|\bar{h}\|^{\cancel{2}}}{\cancel{\|\bar{h}\|}}\right| < \delta+\delta = 2\delta = \varepsilon$$

$|h_1| \leq \|\bar{h}\| \Rightarrow |h_1|^2 = |h_1^2| \leq \|\bar{h}\|^2$

$\therefore$ basta con tomar $\delta = \dfrac{\varepsilon}{2}$, $\therefore f(x,y,z)$ es diferenciable en $(1,1,1)$ ▲

2) Demostrar que $f: \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ es diferenciable en $\bar{x}_0 \in \mathbb{R}^n$ si f_i $i=1,\dots,m$ es diferenciable en $\bar{x}_0$.

Dem.

Como f es diferenciable en $\bar{x}_0 \Rightarrow$ se cumple

$$\lim_{\bar{h}\to\bar{0}} \frac{\|f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h}\|}{\|\bar{h}\|} = 0$$

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$$f(\bar{x}_0+\bar{h}) = \begin{pmatrix} f_1(\bar{x}_0+\bar{h}) \\ f_2(\bar{x}_0+\bar{h}) \\ \vdots \\ f_m(\bar{x}_0+\bar{h}) \end{pmatrix} \quad , \quad f(\bar{x}_0) = \begin{pmatrix} f_1(\bar{x}_0) \\ f_2(\bar{x}_0) \\ \vdots \\ f_m(\bar{x}_0) \end{pmatrix} \quad y$$

$$Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} \partial_{x_1} f_1(\bar{x}_0) \cdots \partial_{x_n} f_1(\bar{x}_0) \\ \partial_{x_1} f_2(\bar{x}_0) \cdots \partial_{x_n} f_2(\bar{x}_0) \\ \vdots \\ \partial_{x_1} f_m(\bar{x}_0) \cdots \partial_{x_n} f_m(\bar{x}_0) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} \partial_{x_j} f_1(\bar{x}_0) h_j \\ \sum_{j=1}^{n} \partial_{x_j} f_2(\bar{x}_0) h_j \\ \vdots \\ \sum_{j=1}^{n} \partial_{x_j} f_m(\bar{x}_0) h_j \end{pmatrix}$$

$$\therefore f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} = \begin{pmatrix} f_1(\bar{x}_0+\bar{h}) - f_1(\bar{x}_0) - \sum_{j=1}^{n} \partial_{x_j} f_1(\bar{x}_0) h_j \\ f_2(\bar{x}_0+\bar{h}) - f_2(\bar{x}_0) - \sum_{j=1}^{n} \partial_{x_j} f_2(\bar{x}_0) h_j \\ \vdots \\ f_m(\bar{x}_0+\bar{h}) - f_m(\bar{x}_0) - \sum_{j=1}^{n} \partial_{x_j} f_m(\bar{x}_0) h_j \end{pmatrix} = \begin{pmatrix} r_1(\bar{h}) \\ r_2(\bar{h}) \\ \vdots \\ r_m(\bar{h}) \end{pmatrix}$$

$$y \quad \| f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} \| = \sqrt{\sum_{\ell=1}^{m} r_\ell^2(\bar{h})}$$

$$\therefore \lim_{\bar{h}\to\bar{0}} \frac{\| f(\bar{x}_0+\bar{h}) - f(\bar{x}_0) - Jf(\bar{x}_0)\bar{h} \|}{\|\bar{h}\|} = \lim_{\bar{h}\to\bar{0}} \frac{\sqrt{\sum_{\ell=1}^{m} r_\ell^2(\bar{h})}}{\sqrt{\sum_{i=1}^{n} h_i^2}}$$

$$= \lim_{\bar{h}\to\bar{0}} \sqrt{\sum_{\ell=1}^{m} \left(\frac{r_\ell^2(\bar{h})}{\sum_{i=1}^{n} h_i^2} \right)} = \lim_{\bar{h}\to\bar{0}} \sqrt{\sum_{\ell=1}^{m} \left(\frac{r_\ell(\bar{h})}{\sqrt{\sum_{i=1}^{n} h_i^2}} \right)^2}$$

$$= \sqrt{\sum_{\ell=1}^{m} \left(\lim_{\bar{h}\to\bar{0}} \frac{r_\ell(\bar{h})}{\sqrt{\sum_{i=1}^{n} h_i^2}} \right)^2} = \sqrt{\sum_{\ell=1}^{m} R_\ell^2} = 0 \quad \text{con } R_\ell = \lim_{\bar{h}\to\bar{0}} \frac{r_\ell(\bar{h})}{\sqrt{\sum_{i=1}^{n} h_i^2}}$$

$\Rightarrow \sum_{\ell=1}^{m} R_\ell^2 = 0$ y ésto es válido solamente si $R_\ell = 0 \quad \forall \ell = 1, \ldots, m$

$\therefore \lim_{\bar{h}\to\bar{0}} \dfrac{r_\ell(\bar{h})}{\|\bar{h}\|} = 0 \quad \forall \ell = 1, \ldots, m$, lo que implica que

$$\lim_{\bar{h}\to\bar{0}} \frac{f_\ell(\bar{x}_0+\bar{h}) - f_\ell(\bar{x}_0) - \sum_{j=1}^{n} \partial_{x_j} f_\ell(\bar{x}_0) h_j}{\|\bar{h}\|} = 0$$ que es precisamente

la condición de que f_ℓ sea diferenciable en $\bar{x}_0$.

$\therefore$ f_ℓ es diferenciable en $\bar{x}_0$ $\forall \ell = 1, \ldots, m$

$\therefore$ si f es diferenciable en $\bar{x}_0 \Rightarrow$ todas las entradas f_i también son diferenciables en $\bar{x}_0$ $\forall i = 1, \ldots, m$ ▲

3) Calcular los siguientes jacobianos.

a) $f(x,y) = (\operatorname{sen}(x+y), xe^{x+y}, x+y)$ en $x_0 = (0,0)$

b) $f(x,y,z) = (xyz + xy, xyz - yz, xyz)$ en $x_0 = (1,1,1)$

c) $f(x,y) = (x^y, y^x)$ en $x_0 = \left(\int_0^\infty e^{-t} dt, \int_0^1 e^2 dt \right)$

d) $f(x,y) = \left(\cos^2 \frac{x}{2} - \operatorname{sen}^2 \frac{x}{2}, \dfrac{\operatorname{sen} xy \cos xy \sec^2 xy}{\tan xy}, \tan y \cot y \operatorname{sen} x \right)$

en $x_0 = (\pi, 0)$

Sol.

a) $$Jf(\bar{x}_0) = \begin{pmatrix} \cos(x+y) & \cos(x+y) \\ (1+x)e^{x+y} & xe^{x+y} \\ 1 & 1 \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix}$$

b) $$Jf(\bar{x}_0) = \begin{pmatrix} yz+y & xz+x & xy \\ yz & xz-z & xy-y \\ yz & xz & xy \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

c) $x_0 = (1, e^2) \Rightarrow$ $$Jf(\bar{x}_0) = \begin{pmatrix} yx^{y-1} & x^y \ln x \\ y^x \ln y & xy^{x-1} \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} e^2 & 0 \\ 2e^2 & 1 \end{pmatrix}$$

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d) $f(x,y)$ se puede reescribir como $f(x,y)=(\cos x, 1, \operatorname{Sen} x)$

pues $\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos^2\frac{x}{2} - \operatorname{Sen}^2\frac{x}{2}$

$$\frac{\operatorname{Sen} xy \cos xy \sec^2 xy}{\tan xy} = \frac{\operatorname{Sen} xy \cos^2 xy}{\operatorname{Sen} xy \cos^2 xy} = 1 \quad y$$

$\tan y \cot y = 1$

$$\Rightarrow Jf(\bar{x}_0) = \begin{pmatrix} -\operatorname{Sen} x & 0 \\ 0 & 0 \\ \cos x & 0 \end{pmatrix}_{\bar{x}_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{pmatrix}$$

4) Regla de la cadena (composición de funciones)

Calcular $J(f \circ g) = Jf(g)\,J(g)$

a) $g(x,y)=(xy, 5x, y^3)$ y $f(x,y,z)=(3x^2+y^2+z^2, 5xyz)$

b) $g(x,y)=(ye^x, xe^y)$ y $f(x,y)=((\ln x)^2, (\ln y)^2)$

Sol.

a) $$Jf(g) = \begin{pmatrix} 6x & 2y & 2z \\ 5yz & 5xz & 5xy \end{pmatrix}_g = \begin{pmatrix} 6(xy) & 2(5x) & 2(y^3) \\ 5(5x)(y^3) & 5(xy)(y^3) & 5(xy)(5x) \end{pmatrix} = \begin{pmatrix} 6xy & 10x & 2y^3 \\ 25xy^3 & 5xy^4 & 25x^2y \end{pmatrix}$$

y $$Jg = \begin{pmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{pmatrix}$$

$$\therefore J(f\circ g) = \begin{pmatrix} 6xy & 10x & 2y^3 \\ 25xy^3 & 5xy^4 & 25x^2y \end{pmatrix}\begin{pmatrix} y & x \\ 5 & 0 \\ 0 & 3y^2 \end{pmatrix} = \begin{pmatrix} 6xy^2+50x & 6x^2y+6y^5 \\ 25xy^4+25xy^4 & 25x^2y^3+75x^2y^3 \end{pmatrix}$$

$$= \begin{pmatrix} 6xy^2+50x & 6x^2y+6y^5 \\ 50xy^4 & 100x^2y^3 \end{pmatrix}$$

b) $Jf(g) = \begin{pmatrix} \frac{2\ln x}{x} & 0 \\ 0 & \frac{2\ln y}{y} \end{pmatrix}_g = \begin{pmatrix} \frac{2\ln(ye^x)}{ye^x} & 0 \\ 0 & \frac{2\ln(xe^y)}{xe^y} \end{pmatrix} = \begin{pmatrix} \frac{2(\ln y + x)e^{-x}}{y} & 0 \\ 0 & \frac{2(\ln x + y)e^{-y}}{x} \end{pmatrix}$

y $Jg = \begin{pmatrix} ye^x & e^x \\ e^y & xe^y \end{pmatrix}$

$\Rightarrow J(f\circ g) = \begin{pmatrix} \frac{2(\ln y + x)e^{-x}}{y} & 0 \\ 0 & \frac{2(\ln x + y)e^{-y}}{x} \end{pmatrix} \begin{pmatrix} ye^x & e^x \\ e^y & xe^y \end{pmatrix} = \begin{pmatrix} 2(x+\ln y) & \frac{2(x+\ln y)}{y} \\ \frac{2(y+\ln x)}{x} & 2(y+\ln x) \end{pmatrix}$

5) Calcula la derivada implícita de las siguientes funciones asumiendo que $y=f(x)$

a) $F(x,y) = x^2+y^2-1$ en $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = p$

b) $F(x,y) = e^{2y+x} + \operatorname{sen}(x^2+y) - 1$ en $(0,0)=p$

c) $F(x,y) = xe^x + ye^y - 2x - 2y$ en $(0,0)=p$

d) $F(x,y) = x^y + y^x - 2xy$ en $(1,1)=p$

$$\frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} \quad \text{con } \frac{\partial F}{\partial y} \neq 0$$

Sol.

a) $\left.\frac{\partial y}{\partial x}\right|_p = -\left.\frac{2x}{2y}\right|_p = -\left.\frac{x}{y}\right|_p = 1$

b) $\left.\frac{\partial y}{\partial x}\right|_p = -\left.\frac{e^{2y+x} + 2x\cos(x^2+y)}{2e^{2y+x} + \cos(x^2+y)}\right|_p = -\frac{1}{3}$

c) $\left.\frac{\partial y}{\partial x}\right|_p = -\left.\frac{(1+x)e^x - 2}{(1+y)e^y - 2}\right|_p = -1$

d) $\left.\frac{\partial y}{\partial x}\right|_p = -\left.\frac{yx^{y-1} + y^x\ln y - 2y}{xy^{x-1} + x^y\ln x - 2x}\right|_p = -1$