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Teorema

Si φ es diferenciable sobre un intervalo $I \subset \mathbb{R}$ y f es diferenciable sobre un intervalo que contiene a $\varphi(I) = \{\varphi(t) \mid t \in I\}$, $\Rightarrow f \circ \varphi$ es diferenciable en I y $(f \circ \varphi)' = f' \circ \varphi \cdot \varphi'$ sobre I.

Dem.

$$(f \circ \varphi)' = ((f_1 \circ \varphi)', \dots, (f_n \circ \varphi)') = (f_1' \circ \varphi \cdot \varphi', \dots, f_n' \circ \varphi \cdot \varphi')$$

$$= (f_1' \circ \varphi, \dots, f_n' \circ \varphi) \cdot \varphi' = f' \circ \varphi \cdot \varphi' \blacktriangle$$

Fórmula de Taylor

Si $f(t) = (f_1(t), \dots, f_n(t))$ es una función vectorial que es continua y diferenciable hasta $n+1$ veces $\Rightarrow$ para cada $i = 1, \dots, n$ se tiene que:

$$f_i(t+h) = f_i(t) + h f_i'(t) + \frac{h^2}{2!} f_i''(t) + \dots + \frac{h^n}{n!} f_i^n(t)$$

$$\therefore f(t+h) = (f_1(t+h), \dots, f_n(t+h))$$

$$= \left(f_1(t) + h f_1'(t) + \frac{h^2}{2!} f_1''(t) + \dots + \frac{h^n}{n!} f_1^n(t), \dots, f_n(t) + h f_n'(t) + \frac{h^2}{2!} f_n''(t) + \dots + \frac{h^n}{n!} f_n^n(t)\right)$$

$$= (f_1(t), \dots, f_n(t)) + h(f_1'(t), \dots, f_n'(t)) + \frac{h^2}{2!}(f_1''(t), \dots, f_n''(t)) + \dots$$

$$+ \frac{h^n}{n!}(f_1^n(t), \dots, f_n^n(t))$$

$$= f(t) + h f'(t) + \frac{h^2}{2!} f''(t) + \dots + \frac{h^n}{n!} f^n(t)$$

$\therefore$ La fórmula de Taylor para funciones vectoriales es:

$$\boxed{f(t+h) = f(t) + h f'(t) + \frac{h^2}{2!} f''(t) + \dots + \frac{h^n}{n!} f^n(t)}$$

Teorema

Si $f(t)=(f_1(t),\ldots,f_n(t))$ es integrable en $[a,b]$, para todo vector $c=(c_1,\ldots,c_n)$ $\Rightarrow$ el producto escalar $c\cdot f$ es integrable en $[a,b]$ y $c\cdot\int_a^b f(t)\,dt=\int_a^b c\cdot f(t)\,dt$

Dem.

$$c\cdot\int_a^b f(t)dt=(c_1,\ldots,c_n)\cdot\left(\int_a^b f_1(t)dt,\ldots,\int_a^b f_n(t)dt\right)$$

$$=c_1\int_a^b f_1(t)dt+\ldots+c_n\int_a^b f_n(t)dt=\int_a^b c_1f_1(t)dt+\ldots+\int_a^b c_nf_n(t)dt$$

$$=\int_a^b (c_1f_1(t)+\ldots+c_nf_n(t))dt=\int_a^b c\cdot f(t)dt \blacktriangle$$

Teorema

Si f y $\|f\|$ son integrables en $[a,b]$ $\Rightarrow \left\|\int_a^b f(t)dt\right\|\leq\int_a^b\|f(t)\|dt$

Dem.

Sea $c=\int_a^b f(t)dt \Rightarrow \|c\|^2=c\cdot c=c\cdot\int_a^b f(t)dt=\int_a^b c\cdot f(t)dt$

$$\leq\int_a^b|c\cdot f(t)|dt=\int_a^b\|c\|\|f(t)\||\cos(c,f)|dt\leq\int_a^b\|c\|\|f(t)\|dt$$

$$\therefore\ \|c\|^{\cancel{2}}\leq\int_a^b\|c\|\|f(t)\|dt=\|\cancel{c}\|\int_a^b\|f(t)\|dt$$

$\Rightarrow \|c\|\leq\int_a^b\|f(t)\|dt$ y como $c=\int_a^b f(t)dt$

$$\Rightarrow \left\|\int_a^b f(t)dt\right\|\leq\int_a^b\|f(t)\|dt \blacktriangle$$

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Ejercicio

Utilizar la definición de derivada para demostrar que:

a) Si $f(t) = a$ con $a \in \mathbb{R}^n$ cte. $\Rightarrow f'(t) = 0$

b) Si $f(t) = a h(t) \Rightarrow f'(t) = a h'(t)$

c) Si $f(t)$ es una función vectorial y $\phi(t)$ es una función escalar

$\Rightarrow (\phi(t) f(t))' = \phi'(t) f(t) + \phi(t) f'(t)$

d) i) $(f+g)'(t) = f'(t) + g'(t)$

ii) $(f(t) \cdot g(t))' = f(t) \cdot g'(t) + g(t) \cdot f'(t)$

iii) $(f(t) \times g(t))' = f(t) \times g'(t) + f'(t) \times g(t)$

e) Sea $r(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ una función diferenciable en t, y t una función diferenciable en s, $\Rightarrow \frac{dr}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds}$

Sol.

a)

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a-a}{\Delta t} = 0$$

b)

$$f'(t) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{a h(t+\Delta t) - a h(t)}{\Delta t} = \lim_{\Delta t \to 0} a \frac{h(t+\Delta t) - h(t)}{\Delta t}$$

$$= a \lim_{\Delta t \to 0} \frac{h(t+\Delta t) - h(t)}{\Delta t} = a h'(t)$$

c)

$$(\phi(t) f(t))' = \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) f(t+\Delta t) - \phi(t) f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) f(t+\Delta t) - \phi(t+\Delta t) f(t) + \phi(t+\Delta t) f(t) - \phi(t) f(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\phi(t+\Delta t)[f(t+\Delta t) - f(t)]}{\Delta t} + \frac{[\phi(t+\Delta t) - \phi(t)] f(t)}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} \phi(t+\Delta t)\right)\left(\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}\right) + \left(\lim_{\Delta t \to 0} \frac{\phi(t+\Delta t) - \phi(t)}{\Delta t}\right)\left(\lim_{\Delta t \to 0} f(t)\right)$$

$$= \phi(t) f'(t) + \phi'(t) f(t)$$

d) i) $$(f+g)'(t) = \lim_{\Delta t \to 0} \frac{(f+g)(t+\Delta t) - (f+g)(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) + g(t+\Delta t) - f(t) - g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} = f'(t) + g'(t) \blacktriangle$$

ii) $$(f(t) \cdot g(t))' = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) \cdot g(t+\Delta t) - f(t) \cdot g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) \cdot g(t+\Delta t) - f(t+\Delta t) \cdot g(t) + f(t+\Delta t) \cdot g(t) - f(t) \cdot g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} f(t+\Delta t) \cdot \frac{[g(t+\Delta t) - g(t)]}{\Delta t} + \lim_{\Delta t \to 0} \frac{[f(t+\Delta t) - f(t)] \cdot g(t)}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} f(t+\Delta t) \right) \cdot \left(\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right) + \left(\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \right) \cdot \left(\lim_{\Delta t \to 0} g(t) \right)$$

$$= f(t) \cdot g'(t) + f'(t) \cdot g(t) \blacktriangle$$

iii) $$(f(t) \times g(t))' = \lim_{\Delta t \to 0} \frac{f(t+\Delta t) \times g(t+\Delta t) - f(t) \times g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) \times g(t+\Delta t) - f(t+\Delta t) \times g(t) + f(t+\Delta t) \times g(t) - f(t) \times g(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t) \times [g(t+\Delta t) - g(t)]}{\Delta t} + \lim_{\Delta t \to 0} \frac{[f(t+\Delta t) - f(t)] \times g(t)}{\Delta t}$$

$$= \left(\lim_{\Delta t \to 0} f(t+\Delta t) \right) \times \left(\lim_{\Delta t \to 0} \frac{g(t+\Delta t) - g(t)}{\Delta t} \right) + \left(\lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} \right) \times \left(\lim_{\Delta t \to 0} g(t) \right)$$

$$= f(t) \times g'(t) + f'(t) \times g(t) \blacktriangle$$

e) $$\frac{dr}{ds} = \frac{df}{ds}\hat{\imath} + \frac{dg}{ds}\hat{\jmath} + \frac{dh}{ds}\hat{k} = \left(\frac{df}{ds}\hat{\imath} + \frac{dg}{ds}\hat{\jmath} + \frac{dh}{ds}\hat{k} \right) \cdot \frac{dt}{dt}$$

$$= \frac{df}{ds} \cdot \frac{dt}{dt}\hat{\imath} + \frac{dg}{ds} \cdot \frac{dt}{dt}\hat{\jmath} + \frac{dh}{ds} \cdot \frac{dt}{dt}\hat{k} = \frac{df}{dt} \cdot \frac{dt}{ds}\hat{\imath} + \frac{dg}{dt} \cdot \frac{dt}{ds}\hat{\jmath} + \frac{dh}{dt} \cdot \frac{dt}{ds}\hat{k}$$

$$= \left(\frac{df}{dt}\hat{\imath} + \frac{dg}{dt}\hat{\jmath} + \frac{dh}{dt}\hat{k} \right) \cdot \frac{dt}{ds} = \frac{dr}{dt} \cdot \frac{dt}{ds} \blacktriangle$$