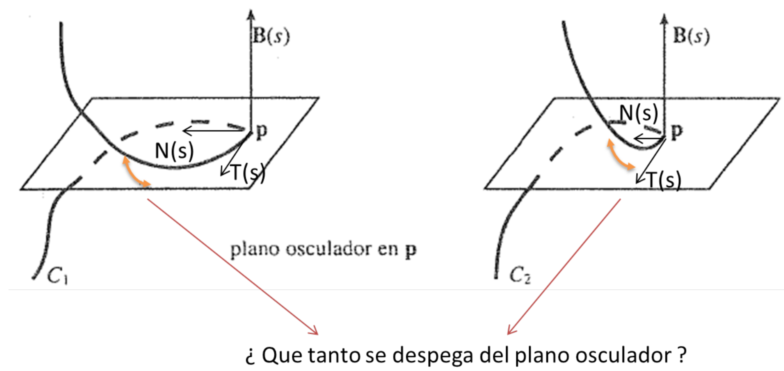
Torsión

Sea $f : I \subset \mathbb{R} \to \mathbb{R}^2$ una curva tres veces diferenciable parametrizada por longitud de arco. Nuestro objetivo consistira en estimar con que rapidez una curva se aleja de su plano osculador



tenemos que

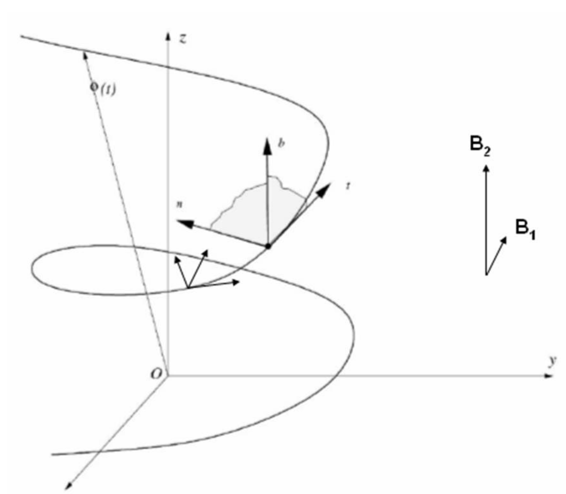
$$\|B\| = 1 \Rightarrow \frac{d\|B\|^2}{ds} \underbrace{=}_{\|B\|^2 = B\cdot B} 0 \Rightarrow \frac{d(B\cdot B)}{ds} = 0 \Rightarrow B\cdot B' + B'\cdot B = 0 \Rightarrow B'\cdot B = 0 \Rightarrow B' \perp B$$

por otro lado

$$B\cdot T = 0 \Rightarrow (B\cdot T)' = 0 \Rightarrow B'\cdot T + T'\cdot B = 0 \underbrace{\Rightarrow}_{T'\cdot B = 0} B'\cdot T = 0 \Rightarrow B' \perp T$$

De lo anterior podemos concluir que B' tiene la dirección del vector N.

Definición 1. *Sea $f : I \subset \mathbb{R} \to \mathbb{R}^3$ una curva tres veces diferenciable parametrizada por longitud de arco tal que $f''(s) \neq 0 \ \forall \ s \in I$. El número $\tau(s)$ tal que $B'(s) = \tau(s)N(s)$ se llama torsión de f en s.*



En la figura anterior vemos que la torsión representa una variación en la dirección del vector binormal, procederemos ahora a desarrollar una fórmula para calcularla
Tenemos que

$$T(s) = f'(s)$$

ahora bien

$$N(s) = \frac{f''(s)}{\|f''(s)\|}$$

por lo tanto

$$N'(s) = \frac{\|f''(s)\|f'''(s) - f''(s)\left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|}\right)}{\|f''(s)\|^2} = \frac{f'''(s)}{\|f''(s)\|} - f''(s)\left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|^3}\right)$$

Luego entonces si

$$B'(s) = T(s) \times N'(s)$$

se tiene que

$$B'(s) = f'(s)\times\frac{f'''(s)}{\|f''(s)\|} - f''(s)\left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|^3}\right) = \frac{1}{\|f''(s)\|}f'(s)\times f'''(s) - \left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|^3}\right)f'(s)\times f''(s)$$

La torsión esta dada por

$$B'(s) = \tau(s)N(s) \Rightarrow B'(s)\cdot N(s) = \tau(s)N(s)\cdot N(s) \Rightarrow B'(s)\cdot N(s) = \tau(s)\|N(s)\|^2 \Rightarrow B'(s)\cdot N(s) = |\tau(s)|$$

Vamos a calcular $B'(s)\cdot N(s)$ tenemos que

$$\left(\frac{1}{\|f''(s)\|}f'(s)\times f'''(s) - \left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|^3}\right)f'(s)\times f''(s)\right)\cdot\frac{f''(s)}{\|f''(s)\|} =$$

$$\frac{1}{\|f''(s)\|^2}f'(s)\times f'''(s)\cdot f''(s) - \cancel{\left(\frac{f''(s)\cdot f'''(s)}{\|f''(s)\|^4}\right)f'(s)\times f''(s)\cdot f''(s)} = \frac{1}{\|f''(s)\|^2}f'(s)\times f'''(s)\cdot f''(s)$$

La cancelación es porque

$$f'(s)\times f''(s)\cdot f''(s) = 0$$

y como

$$k(s) = \|f''\|$$

se tiene entonces que

$$\tau(s) = \frac{f'(s)\times f'''(s)\cdot f''(s)}{k(s)^2} = -\frac{f'(s)\times f''(s)\cdot f'''(s)}{k(s)^2}$$

Ahora vamos a expresar la torsión en términos de t, se tiene que Ya hemos visto que

$$T(s) = f'(s) = \frac{f'(t)}{\|f'(t)\|}$$

Por lo tanto

$$T'(s) = f''(s) = \frac{d\left(\frac{f'(t)}{\|f'(t)\|}\right)}{ds} = \frac{d\left(\frac{f'(t)}{\|f'(t)\|}\right)}{dt}\frac{dt}{ds} = \frac{d\left(\frac{f'(t)}{\|f'(t)\|}\right)}{dt}\frac{1}{\frac{ds}{dt}} = \frac{d\left(\frac{f'(t)}{\|f'(t)\|}\right)}{dt}\varphi'(s) =$$

$$\frac{d\left(\frac{f'(t)}{\|f'(t)\|}\right)}{dt}\frac{1}{\|f'(t)\|} = \left(\frac{\|f'(t)\|f'' - f'(t)\frac{d(\|f'(t)\|)}{dt}}{\|f'(t)\|^2}\right)\frac{1}{\|f'(t)\|} \underbrace{=}_{*} \left(\frac{\|f'(t)\|f'' - f'(t)\frac{f'\cdot f''}{\|f'(t)\|}}{\|f'(t)\|^2}\right)\frac{1}{\|f'(t)\|}$$

Donde *

$$\frac{d\left(\|f'(t)\|\right)}{dt} = \frac{d\sqrt{f'(t)\cdot f'(t)}}{dt} = \frac{f'(t)\cdot f''(t)}{\|f'(t)\|}$$

Por tanto

$$\left(\frac{\|f'(t)\|f'' - f'(t)\frac{f'\cdot f''}{\|f'(t)\|}}{\|f'(t)\|^2}\right)\frac{1}{\|f'(t)\|} = \left(\frac{\|f'(t)\|^2 f'' - (f'\cdot f'')\, f'}{\|f'(t)\|}\right)\frac{1}{\|f'(t)\|^3} = \frac{1}{\|f'\|^4}\left(\|f'(t)\|^2 f'' - (f'\cdot f'')\, f'\right)$$

Por lo tanto

$$f''(s) = \frac{1}{\|f'\|^4}\left(\|f'(t)\|^2 f''(t) - (f'(t)\cdot f''(t))\, f'(t)\right) \tag{1}$$

$$f'(s)\times f''(s) = \frac{f'(t)}{\|f'(t)\|}\times\left(\frac{1}{\|f'\|^4}\left(\|f'(t)\|^2 f''(t) - (f'(t)\cdot f''(t))\, f'(t)\right)\right) =$$

$$\left(\frac{1}{\|f'(t)\|^3}\right) f'(t)\times f''(t) - \left(\frac{f'(t)\cdot f''(t)}{\|f'(t)\|}\right)\cancel{f'(t)\times f'(t)} = \left(\frac{1}{\|f'(t)\|^3}\right) f'(t)\times f''(t)$$

Mientras que

$$f'''(s) = \frac{df''(s)}{ds} = \left(\frac{df''(s)}{dt}\right)\frac{dt}{ds} = \left(\frac{df''(s)}{dt}\right)\frac{1}{\frac{ds}{dt}} = \left(\frac{df''(s)}{dt}\right)\frac{1}{\|f'(t)\|} =$$

$$\frac{1}{\|f'(t)\|}\frac{d}{dt}\left(\frac{1}{\|f'\|^4}\left(\|f'(t)\|^2 f''(t) - (f'(t)\cdot f''(t))\, f'(t)\right)\right) =$$

$$\left(\frac{1}{\|f'(t)\|}\right)\left[\left(\frac{1}{\|f'(t)\|^2}f'''(t)\right) + \left(\frac{1}{\|f'(t)\|}\right)' f''(t) - \left(\frac{f'(t)\cdot f''}{\|f'(t)\|^4}\right) f''(t) - f'(t)\left(\frac{f'(t)\cdot f''}{\|f'(t)\|^4}\right)\right]$$

Por lo tanto hacemos

$$f'(s)\times f''(s)\cdot f'''(s) =$$

$$\left(\frac{1}{\|f'(t)\|^3}\right) f'(t)\times f''(t)\cdot\left(\frac{1}{\|f'(t)\|}\right)\left[\left(\frac{1}{\|f'(t)\|^2}f'''(t)\right) + \cancel{\left(\frac{1}{\|f'(t)\|}\right)' f''(t)} - \cancel{\left(\frac{f'(t)\cdot f''}{\|f'(t)\|^4}\right) f''(t)} - \cancel{f'(t)\left(\frac{f'(t)\cdot f''}{\|f'(t)\|^4}\right)}\right]$$

$$= \frac{1}{\|f'(t)\|^3} f'(t) \times f''(t) \cdot \frac{1}{\|f'(t)\|^3} f'''(t) = \frac{1}{\|f'(t)\|^6} f'(t) \times f''(t) \cdot f'''$$

de la igualdad

$$\frac{f'(s) \times f'''(s) \cdot f''(s)}{k(s)^2} = -\frac{f'(s) \times f''(s) \cdot f'''(s)}{k(s)^2} = -\frac{1}{\|f'(t)\|^6} \frac{f'(t) \times f''(t) \cdot f'''}{\left(\frac{\|f' \times f''\|}{\|f'(t)\|^3}\right)^2}$$

Tenemos que la torsión esta dada por

$$\tau(t) = -\frac{f'(t) \times f''(t) \cdot f'''}{\left(\|f'(t) \times f''(t)\|\right)^2}$$