Teorema de la Función Implícita ($f : \mathbb{R} \to \mathbb{R}$)

Teorema 1. *Considere la función $y = f(x)$. Sea $(x_0, y_0) \in \mathbb{R}^2$ un punto tal que $F(x_0, y_0) = 0$. Suponga que la función F tiene derivadas parciales continuas en alguna bola con centro (x_0, y_0) y que $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$.*
Entonces $F(x, y) = 0$ se puede resolver para y en términos de x y definir así una función $y = f(x)$ con dominio en una vecindad de (x_0, y_0), tal que $y_0 = f(x_0)$, lo cual tiene derivadas continuas en $\mathcal{V}$ que pueden calcularse como $y' = f'(x) = -\dfrac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}$, $x \in \mathcal{V}$.

Ejercicio Si

$$y' = f'(x) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}$$

calcular y''

Solución En este caso

$$y'' = -\frac{\left(\frac{\partial F}{\partial y}\right)\left[\frac{\partial^2 F}{\partial x^2}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y \partial x}\frac{dy}{dx}\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial x \partial y}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y^2}\frac{dy}{dx}\right]}{\left(\frac{\partial F}{\partial y}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial y}\right)\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y \partial x}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial x \partial y}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y^2}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}\right)\right]}{\left(\frac{\partial F}{\partial y}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial y}\right)^2\left(\frac{\partial^2 F}{\partial x^2}\right) - \left(\frac{\partial^2 F}{\partial y \partial x}\right)\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right) - \left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right)\left(\frac{\partial^2 F}{\partial x \partial y}\right) + \left(\frac{\partial F}{\partial x}\right)^2\left(\frac{\partial^2 F}{\partial y^2}\right)}{\left(\frac{\partial F}{\partial y}\right)^3}$$

$$= -\frac{\left(\frac{\partial F}{\partial y}\right)^2\left(\frac{\partial^2 F}{\partial x^2}\right) - 2\left(\frac{\partial^2 F}{\partial y \partial x}\right)\left(\frac{\partial F}{\partial x}\right)\left(\frac{\partial F}{\partial y}\right) + \left(\frac{\partial F}{\partial x}\right)^2\left(\frac{\partial^2 F}{\partial y^2}\right)}{\left(\frac{\partial F}{\partial y}\right)^3}$$

Teorema de la Función Implícita ($f : \mathbb{R}^2 \to \mathbb{R}$)

Teorema 2. *Considere la función $F(x, y, z)$. Sea $(x_0, y_0, z_0) \in \mathbb{R}^3$ un punto tal que $F(x_0, y_0, z_0) = 0$. Suponga que la función F tiene derivadas parciales $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$, $\frac{\partial F}{\partial z}$ continuas en alguna bola con centro (x_0, y_0, z_0) y que $\frac{\partial F}{\partial z}(x_0, y_0, z_0) \neq 0$.*
Entonces $F(x, y, z) = 0$ se puede resolver para z en términos de x, y y definir así una función $z = f(x, y)$ con dominio en una vecindad de (x_0, y_0, z_0), tal que $z_0 = f(x_0, y_0)$, lo cual tiene derivadas continuas en $\mathcal{V}$ que pueden calcularse como

$$\frac{dz}{dx}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial z}(x,y)} \qquad \frac{dz}{dy}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial z}(x,y)}$$

Ejercicio Si

$$\frac{dz}{dx}(x,y) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial z}(x,y)}$$

calcular

$$\frac{\partial^2 F}{\partial x^2}$$

Solución tenemos que

$$\frac{\partial^2 F}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial z}(x,y)}\right) = -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial x^2}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y \partial x}\frac{dy}{dx} + \frac{\partial^2 F}{\partial z \partial x}\frac{dz}{dx}\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial x \partial z}\frac{dx}{dx} + \frac{\partial^2 F}{\partial y \partial z}\frac{dy}{dx} + \frac{\partial^2 F}{\partial z^2}\frac{dz}{dx}\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z \partial x}\frac{dz}{dx}\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial x \partial z} + \frac{\partial^2 F}{\partial z^2}\frac{dz}{dx}\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial z \partial x}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right)\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial x \partial z} + \frac{\partial^2 F}{\partial z^2}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right)\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial^2 F}{\partial x^2} - 2\frac{\partial^2 F}{\partial z \partial x}\frac{\partial F}{\partial x}\frac{\partial F}{\partial z} + \left(\frac{\partial F}{\partial x}\right)^2\frac{\partial^2 F}{\partial z^2}}{\left(\frac{\partial F}{\partial z}\right)^3}$$

Ejercicio Si

$$\frac{dz}{dy}(x,y) = -\frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial z}(x,y)}$$

calcular

$$\frac{\partial^2 F}{\partial y^2}$$

Solución tenemos que

$$\frac{\partial^2 F}{\partial y^2} = \frac{\partial}{\partial y}\left(-\frac{\frac{\partial F}{\partial y}(x,y)}{\frac{\partial F}{\partial z}(x,y)}\right) = -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial y\partial x}\frac{dx}{dy} + \frac{\partial^2 F}{\partial y^2}\frac{dy}{dy} + \frac{\partial^2 F}{\partial z\partial y}\frac{dz}{dy}\right] - \left(\frac{\partial F}{\partial y}\right)\left[\frac{\partial^2 F}{\partial x\partial z}\frac{dx}{dy} + \frac{\partial^2 F}{\partial y\partial z}\frac{dy}{dy} + \frac{\partial^2 F}{\partial z^2}\frac{dz}{dy}\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z\partial y}\frac{dz}{dy}\right] - \left(\frac{\partial F}{\partial y}\right)\left[\frac{\partial^2 F}{\partial y\partial z} + \frac{\partial^2 F}{\partial z^2}\frac{dz}{dy}\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z\partial y}\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right)\right] - \left(\frac{\partial F}{\partial y}\right)\left[\frac{\partial^2 F}{\partial y\partial z} + \frac{\partial^2 F}{\partial z^2}\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right)\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$= -\frac{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial^2 F}{\partial y^2} - 2\frac{\partial^2 F}{\partial z\partial y}\frac{\partial F}{\partial y}\frac{\partial F}{\partial z} + \left(\frac{\partial F}{\partial y}\right)^2\frac{\partial^2 F}{\partial z^2}}{\left(\frac{\partial F}{\partial z}\right)^3}$$

Ejercicio Si

$$\frac{dz}{dy}(x,y) = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

calcular

$$\frac{\partial^2 F}{\partial y\partial x}$$

Solución tenemos que

$$\frac{\partial^2 F}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\right) =$$

$$-\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial y\partial x} + \frac{\partial^2 F}{\partial z\partial x}\frac{\partial z}{\partial y}\right] - \left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial y\partial z} + \frac{\partial^2 F}{\partial z^2}\frac{\partial z}{\partial y}\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$-\frac{\left(\frac{\partial F}{\partial z}\right)\left[\frac{\partial^2 F}{\partial y\partial x}+\frac{\partial^2 F}{\partial z\partial x}\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right)\right]-\left(\frac{\partial F}{\partial x}\right)\left[\frac{\partial^2 F}{\partial y\partial z}+\frac{\partial^2 F}{\partial z^2}\left(-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\right)\right]}{\left(\frac{\partial F}{\partial z}\right)^2}$$

$$=-\frac{\left(\frac{\partial F}{\partial z}\right)^2\frac{\partial^2 F}{\partial y\partial x}-\frac{\partial^2 F}{\partial z\partial x}\frac{\partial F}{\partial y}\frac{\partial F}{\partial z}-\frac{\partial^2 F}{\partial y\partial z}\frac{\partial F}{\partial x}\frac{\partial F}{\partial z}+\frac{\partial^2 F}{\partial z^2}\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}}{\left(\frac{\partial F}{\partial z}\right)^3}$$

Teorema de la Función Implicita (version sistemas de ecuaciones)

Consideremos ahora el sistema

$au+bv-k_1x=0$
$cu+dv-k_2y=0$

con a,b,c,d,k_1,k_2 constantes. Nos preguntamos cuando podemos resolver el sistema para u y v en términos de x y y. Si escribimos el sistema como

$au+bv=k_1x$
$cu+dv=k_2y$

y sabemos que este sistema tiene solución si $det\begin{vmatrix} a & b \\ c & d \end{vmatrix}\neq 0$ en tal caso escribimos

$$u=\frac{1}{det\begin{vmatrix} a & b \\ c & d \end{vmatrix}}(k_1dx-k_2by), \qquad v=\frac{1}{det\begin{vmatrix} a & b \\ c & d \end{vmatrix}}(k_2ay-k_1cx).$$

Esta solución no cambiaria si consideramos

$au+bv=f_1(x,y)$
$cu+dy=f_2(x,y)$

donde f_1 y f_2 son funciones dadas de x y y. La posibilidad de despejar las variables u y v en términos de x y y recae sobre los coeficientes de estas variables en las ecuaciones dadas.

Ahora si consideramos ecuaciones no lineales en u y v escribimos el sistema como

$g_1(u,v)=f_1(x,y)$
$g_2(u,v)=f_2(x,y)$

nos preguntamos cuando del sistema podemos despejar a uy v en términos de x y y. Mas generalmente, consideramos el problema siguiente, dadas las funciones F y G de las variables u,v,x,y nos preguntamos cuando de las expresiones

$F(x,y,u,v)=0$
$G(x,y,u,v)=0$

podemos despejar a u y v en términos de x y y en caso de ser posible diremos que las funciones $u=\varphi_1(x,y)$ y $v=\varphi_2(x,y)$ son funciones implícitas dadas. Se espera que $\exists'$n funciones $u=\varphi_1(x,y)$ y $v=\varphi_2(x,y)$ en

$F(x,y,\varphi_1(x,y),\varphi_2(x,y)$
$G(x,y,\varphi_1(x,y),\varphi_2(x,y)$

con (x,y) en alguna vecindad V.

Suponiendo que existen φ_1 y φ_2 veamos sus derivadas

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial F}{\partial x}$$

$$\frac{\partial G}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial G}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=0 \quad\Rightarrow\quad \frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=-\frac{\partial G}{\partial x}$$

Lo anterior se puede ver como un sistema de 2 ecuaciones con 2 incógnitas $\frac{\partial u}{\partial x}$ y $\frac{\partial v}{\partial x}$. Aquí se ve que para que el sistema tenga solución

$$det\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}\neq 0 \text{ en } (P) \text{ (el } det \text{ Jacobiano) y según la regla de Cramer}$$

$$\frac{\partial u}{\partial x}=-\frac{\det\begin{vmatrix}-\frac{\partial F}{\partial x} & \frac{\partial F}{\partial v}\\ -\frac{\partial G}{\partial x} & \frac{\partial G}{\partial v}\end{vmatrix}}{\det\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}},\qquad \frac{\partial v}{\partial x}=-\frac{\det\begin{vmatrix}\frac{\partial F}{\partial u} & -\frac{\partial F}{\partial x}\\ \frac{\partial G}{\partial u} & -\frac{\partial G}{\partial x}\end{vmatrix}}{det\begin{vmatrix}\frac{\partial F}{\partial u} & \frac{\partial F}{\partial v}\\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v}\end{vmatrix}}\qquad \text{(con los dos } det \text{ Jacobianos).}$$

Analogamente si derivamos con respecto a y obtenemos

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial y}=-\frac{\partial F}{\partial y}$$

$$\frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial G}{\partial y}$$

de donde

$$\frac{\partial u}{\partial y} = -\frac{\det \begin{vmatrix} -\frac{\partial F}{\partial y} & \frac{\partial F}{\partial v} \\ -\frac{\partial G}{\partial y} & \frac{\partial G}{\partial v} \end{vmatrix}}{det \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}}, \qquad \frac{\partial v}{\partial y} = -\frac{\det \begin{vmatrix} \frac{\partial F}{\partial u} & -\frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial u} & -\frac{\partial G}{\partial y} \end{vmatrix}}{det \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}} \qquad \text{(con los dos } det \text{ Jacobianos).}$$

Al determinante $det \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix}$ lo llamamos Jacobiano y lo denotamos por $\frac{\partial(F,G)}{\partial(u,v)}$.

Teorema de la Función Implícita (sistemas de ecuaciones)

Teorema 3. *Considere las funciones $z_1 = F(x,y,u,v)$ y $z_2 = G(x,y,u,v)$. Sea $P = (x,y,u,v) \in \mathbb{R}^4$ un punto tal que $F(P) = G(P) = 0$. Suponga que en una bola $B \in \mathbb{R}^4$ de centro P las funciones F y G tienen (sus cuatro) derivadas parciales continuas. Si el Jacobiano $\frac{\partial(F,G)}{\partial(u,v)}(P) \neq 0$ entonces las expresiones $F(x,y,u,v) = 0$ y $G(x,y,u,v) = 0$ definen funciones (implícitas) $u = \varphi_1(x,y)$ y $v = \varphi_2(x,y)$ definidas en una vecindad v de (x,y) las cuales tienen derivadas parciales continuas en v que se pueden calcular como se menciona arriba.*

Demostración. Dado que

$$det \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial F}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0$$

entonces $\frac{\partial F}{\partial u}(p)$, $\frac{\partial F}{\partial v}(p)$, $\frac{\partial G}{\partial u}(p)$, $\frac{\partial G}{\partial v}(p)$ no son cero al mismo tiempo, podemos suponer sin perdida de generalidad que $\frac{\partial G}{\partial v}(p) \neq 0$. Entonces la función $z_1 = G(x,y,u,v)$ satisface las hipótesis del T.F.I y en una bola abierta con centro p, v se puede escribir como $v = \psi(x,y,u)$.
Hacemos ahora

$$H(x,y,u) = F(x,y,u,\psi(x,y,u))$$

y tenemos que

$$\frac{\partial H}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial u} + \frac{\partial F}{\partial v}\frac{\partial \psi}{\partial u} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\frac{\partial \psi}{\partial u}$$

por otro lado

$$\frac{\partial \psi}{\partial u} = -\frac{\frac{\partial G}{\partial u}}{\frac{\partial G}{\partial v}}$$

por lo tanto

$$\frac{\partial H}{\partial u} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\frac{\partial \psi}{\partial u} = \frac{\partial F}{\partial u} + \frac{\partial F}{\partial v}\left(-\frac{\frac{\partial G}{\partial u}}{\frac{\partial G}{\partial v}}\right) = \frac{\frac{\partial F}{\partial u}\frac{\partial G}{\partial v} - \frac{\partial F}{\partial v}\frac{\partial G}{\partial u}}{\frac{\partial G}{\partial v}} \neq 0$$

por lo tanto para $H(x,y,u) = 0$ tenemos que existe una función $u = \varphi_1(x,y)$ y por lo tanto $v = \psi(x,y,u) = \psi(x,y,\varphi_1(x,y,u)) = \varphi_2(x,y)$ y por tanto u, v se pueden expresar en términos de x, y en una vecindad de p. $\square$