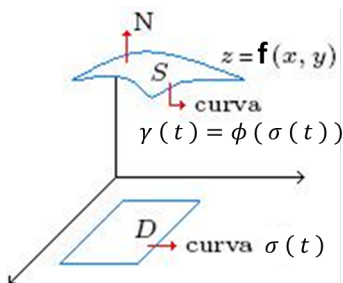


Teorema de Stokes



El teorema de Stokes relaciona la integral de línea de un campo vectorial alrededor de una curva cerrada simple $\gamma \in \mathbb{R}^3$, con la integral sobre una superficie de la cual γ es la frontera. Es decir, si se tiene S una superficie orientada con vector normal unitario N y frontera una curva cerrada γ y un campo vectorial F de clase C^1 se cumple que

$$\int_{\gamma} F = \iint_S (\text{rot}F \cdot N) dS$$

Si la superficie S es la gráfica de una función $z = f(x, y)$ con (x, y) variando en una región D del plano xy , que tiene derivadas parciales segundas continuas, podemos parametrizarla por medio de la función $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ dada por

$$\phi(u, v) = (u, v, f(u, v))$$

Mientras que el campo $F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ esta dado por

$$F(x, y, z) = P(x, y, z)\hat{i} + Q(x, y, z)\hat{j} + R(x, y, z)\hat{k}$$

donde P, Q y R tienen parciales primeras continuas

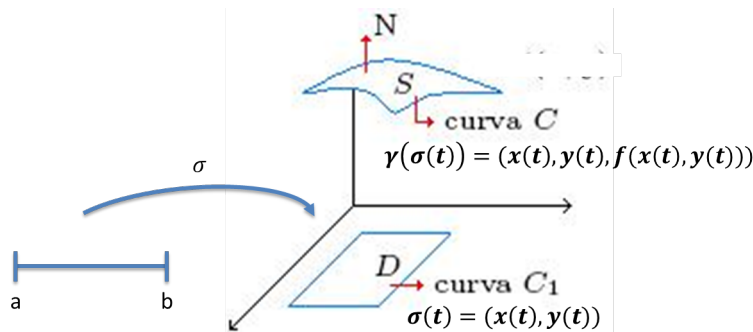
Ahora para la frontera de la superficie S consideramos la curva $\sigma : \mathbb{R} \rightarrow \mathbb{R}^2$ dada por

$$\sigma(t) = (x(t), y(t))$$

de esta manera la frontera de la superficie S sera la función $\gamma : \mathbb{R} \rightarrow \mathbb{R}^3$ dada por

$$\gamma(t) = \phi(\sigma(t)) = \phi(x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j} + z(t) = f(x(t), y(t))\hat{k}$$

con $a \leq t \leq b$



tenemos entonces que

$$\int_{\gamma} F = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt = \int_a^b P(\gamma(t), Q(\gamma(t)), R(\gamma(t))) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right) dt =$$

$$\begin{aligned} & \int_a^b \left[P(\gamma(t)) \frac{dx}{dt} + Q(\gamma(t)) \frac{dy}{dt} + R(\gamma(t)) \left(\frac{dz}{dx} \frac{dx}{dt} + \frac{dz}{dy} \frac{dy}{dt} \right) \right] dt \\ &= \int_a^b \left[\left(P(\gamma(t)) + R(\gamma(t)) \frac{\partial z}{\partial x} \right) \frac{dx}{dt} + \left(Q(\gamma(t)) + R(\gamma(t)) \frac{\partial z}{\partial y} \right) \frac{dy}{dt} \right] dt = \\ & \int_{\gamma} \left(P(\gamma(t)) + R(\gamma(t)) \frac{\partial z}{\partial x} \right) dx + \left(Q(\gamma(t)) + R(\gamma(t)) \frac{\partial z}{\partial y} \right) dy \stackrel{\text{Green}}{=} \\ & \int \int_D \left[\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) \right] dA = \end{aligned}$$

Al ser $u = x$ y $v = y$, se tiene que $z = f(x, y) = f(u, v)$ se tiene

$$\begin{aligned} P = P(x, y, z) = P(u, v, z) &\Rightarrow_{\text{Regla Cadena}} \frac{\partial P}{\partial y} = \frac{\partial P}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial P}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} \\ Q = Q(x, y, z) = Q(u, v, z) &\Rightarrow_{\text{Regla Cadena}} \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} \\ R = R(x, y, z) = R(u, v, z) &\Rightarrow_{\text{Regla Cadena}} \frac{\partial R}{\partial x} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \\ R = R(x, y, z) = R(u, v, z) &\Rightarrow_{\text{Regla Cadena}} \frac{\partial R}{\partial y} = \frac{\partial R}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial R}{\partial v} \frac{\partial v}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \end{aligned}$$

por lo tanto

$$\begin{aligned} \frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) &= \frac{\partial Q}{\partial x} + R \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} \frac{\partial R}{\partial x} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z} \frac{\partial z}{\partial x} + R \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} \frac{\partial R}{\partial x} + \frac{\partial z}{\partial y} \frac{\partial R}{\partial z} \frac{\partial z}{\partial x} \\ \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) &= \frac{\partial P}{\partial y} + R \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} \frac{\partial R}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} \frac{\partial z}{\partial y} + R \frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x} \frac{\partial R}{\partial y} + \frac{\partial z}{\partial x} \frac{\partial R}{\partial z} \frac{\partial z}{\partial y} \end{aligned}$$

Por tener $z = f(x, y)$ se tiene $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$; entonces

$$\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) = - \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \left(\frac{\partial z}{\partial x} \right) - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

por lo tanto

$$\begin{aligned} & \int \int_D \left[\frac{\partial}{\partial x} \left(Q + R \frac{\partial z}{\partial y} \right) - \frac{\partial}{\partial y} \left(P + R \frac{\partial z}{\partial x} \right) \right] dA = \\ & \int \int_D \left[- \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \left(\frac{\partial z}{\partial x} \right) - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \left(\frac{\partial z}{\partial y} \right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right] dA \end{aligned}$$

Calculamos ahora $\int \int_S \text{rot}F \cdot NdS$

$$\text{rot}F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)$$

$$N = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right)$$

$$\begin{aligned} \therefore \text{rot}F \cdot N &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot \left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right) = \\ &= -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{aligned}$$

$$\therefore \int \int_D \text{rot}F \cdot N dS = \int \int_D \left(-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\therefore \int \int_S \text{rot}F \cdot N dS = \int_{\gamma} F \cdot dR$$

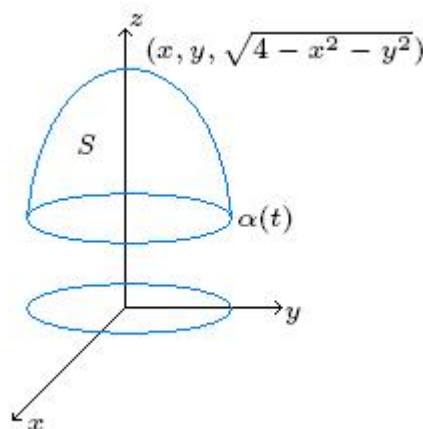
Ejemplo: Comprobar Stokes calculando el flujo del rotacional del campo

$$F(x, y, z) = (y - 2x, yz^2, -y^2z) \quad S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 4, z > 1\}$$

Sol. Parametrizamos el casquete

$$r : D \subset \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad (x, y) = (x, y, \sqrt{4 - x^2 - y^2})$$

de esta forma $S = r(D)$ donde $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 3\}$



se tiene que el vector normal es

$$N(x, y) = \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \left(\frac{x}{\sqrt{4 - x^2 - y^2}}, \frac{y}{\sqrt{4 - x^2 - y^2}}, 1 \right)$$

$$\text{Y el rotacional es } \operatorname{rot} F = \nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-2x & yz^2 & -y^2z \end{vmatrix} = (-4yz, 0, -1)$$

$$\begin{aligned} \therefore \int_S \operatorname{rot} F \cdot N dx dy &= \int_D (-4xy - 1) dx dy = \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} (-4xy - 1) dx dy \int_{-\sqrt{3}}^{\sqrt{3}} -2x^2y - x \Big|_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} dy = \\ &= \int_{-\sqrt{3}}^{\sqrt{3}} -2(3-y^2)y - \sqrt{3-y^2} - [-2(-\sqrt{3-y^2})] y - \sqrt{3-y^2} dy = \int_{-\sqrt{3}}^{\sqrt{3}} -2\sqrt{3-y^2} dy = -2 \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{3-y^2} dy = \\ &= -2\pi \frac{(\sqrt{3})^2}{2} = -3\pi \end{aligned}$$

Para comprobar Stokes calculamos la integral curvilínea.

$$\int_{\alpha} (y-2x)dx + yz^2dy - y^2zdz \quad \text{Siendo } \alpha = r \circ \gamma, \quad \gamma(t) = (\sqrt{3}\cos(t), \sqrt{3}\sen(t))$$

$$\therefore \alpha : [0, 2\pi] \rightarrow \mathbb{R}^3 \text{ con } \alpha(t) = r(\gamma(t)) = (\sqrt{3}\cos(t), \sqrt{3}\sen(t), 1)$$

\therefore la integral vale

$$\begin{aligned} \int_{\alpha} (y-2x)dx + yz^2dy - y^2zdz &= \int_0^{2\pi} F(\alpha(t)) \cdot \alpha'(t) dt = \\ &= \int_0^{2\pi} (\sqrt{3}\sen(t) - 2\sqrt{3}\cos(t), \sqrt{3}\sen(t), -3\sen^2(t)) \cdot (-\sqrt{3}\sen(t), \sqrt{3}\cos(t), 0) dt = \\ &= \int_0^{2\pi} (-3\sen^2(t) + 9\sen(t)\cos(t)) dt = -3\pi \end{aligned}$$