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Repaso de integrales

Métodos de integración:

1) Cambio de variable
2) Por partes
3) Sustitución trigonométrica
4) Fracciones parciales

Ejemplos

① $\int \frac{(1+e^x)\,dx}{1-e^x}$

Para integrar esto sumamos un cero.

$$\int \frac{(1+e^x)\,dx}{1-e^x} = \int \frac{(1+e^x+e^x-e^x)\,dx}{1-e^x} = \int \left(1+\frac{2e^x}{1-e^x}\right)dx = x+\int \frac{2e^x}{1-e^x}\,dx \cdots \circledast$$

sea $u=1-e^x \Rightarrow du=-e^x dx \quad \therefore \quad \circledast = x-2\int \frac{du}{u} = x-2\ln u$

$$\therefore \int \frac{(1+e^x)\,dx}{1-e^x} = x-2\ln(1-e^x)+C$$

② $\int x\sqrt{x+1}\,dx$

Hacemos un cambio de variable $x+1=u^2 \Rightarrow x=u^2-1 \Rightarrow dx=2u\,du$

$$\Rightarrow \int x\sqrt{x+1}\,dx = \int (u^2-1)\sqrt{u^2}\,2u\,du = 2\int (u^4-u^2)\,du = 2\left(\frac{u^5}{5}-\frac{u^3}{3}\right)$$

$$\therefore \int x\sqrt{x+1}\,dx = 2\left(\frac{(x+1)^{5/2}}{5}-\frac{(x+1)^{3/2}}{3}\right)+C$$

③ $\int \cos^2 x\,dx$

Recordando que $\cos 2x = \cos(x+x) = \cos x\cos x - \operatorname{sen} x \operatorname{sen} x$

$$= \cos^2 x - \operatorname{sen}^2 x = \cos^2 x - (1-\cos^2 x)$$

$$= 2\cos^2 x - 1$$

$$\Rightarrow \cos^2 x = \frac{\cos 2x + 1}{2}$$

$$\therefore \int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx = \frac{x}{2} + \frac{1}{2}\int \cos 2x\,dx \;\cdots \bigstar$$

haciendo $u = 2x \Rightarrow du = 2dx$

$$\Rightarrow \bigstar = \frac{x}{2} + \frac{1}{4}\int \cos u\,du = \frac{x}{2} + \frac{\operatorname{sen} u}{4}$$

$$\therefore \int \cos^2 x\,dx = \frac{x}{2} + \frac{\operatorname{sen} 2x}{4} + C$$

④ $\int e^x \operatorname{sen} x\,dx$

Procedemos por partes, sea $u = e^x$ $\quad dv = \operatorname{sen} x\,dx$
$du = e^x dx \quad v = -\cos x$

$$\Rightarrow \int e^x \operatorname{sen} x\,dx = -e^x \cos x + \int e^x \cos x\,dx \;\cdots \text{I}$$

Procedemos otra vez por partes, sea $u = e^x$ $\quad dv = \cos x\,dx$
$du = e^x dx \quad v = \operatorname{sen} x$

$$\Rightarrow \text{I} = -e^x \cos x + e^x \operatorname{sen} x - \int e^x \operatorname{sen} x\,dx$$

$$\Rightarrow 2\int e^x \operatorname{sen} x\,dx = e^x(\operatorname{sen} x - \cos x)$$

$$\therefore \int e^x \operatorname{sen} x\,dx = \frac{e^x}{2}(\operatorname{sen} x - \cos x) + C$$

⑤ $$\int \frac{dx}{\sqrt{4x^2+8x+8}} = \frac{1}{2}\int \frac{dx}{\sqrt{x^2+2x+2}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x^2+2x+1)+1}} = \frac{1}{2}\int \frac{dx}{\sqrt{(x+1)^2+1}} \;\cdots \triangle$$

sea $u = x+1 \Rightarrow du = dx$

$\Rightarrow \triangle = \frac{1}{2}\int \frac{du}{\sqrt{u^2+1}}$ haciendo una sustitución trigonométrica

$u = \tan w \Rightarrow du = \sec^2 w\,dw$, por lo que la $\int$ resulta en

$$\frac{1}{2}\int \frac{du}{\sqrt{u^2+1}} = \frac{1}{2}\int \frac{\sec^2 w\,dw}{\sqrt{\tan^2 w+1}} = \frac{1}{2}\int \frac{\sec^2 w\,dw}{\sqrt{\sec^2 w}} = \frac{1}{2}\int \sec w\,dw$$

$$= \frac{1}{2}\int \sec w\left(\frac{\sec w + \tan w}{\sec w + \tan w}\right)dw = \frac{1}{2}\int \frac{\sec^2 w + \sec w \tan w}{\sec w + \tan w}\,dw \;\cdots ⊙$$

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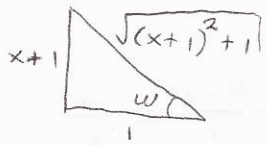
haciendo $v = \sec w + \tan w \Rightarrow dv = (\sec w \tan w + \sec^2 w)\,dw$

$\Rightarrow \odot = \frac{1}{2}\int \frac{dv}{v} = \frac{1}{2}\ln v = \frac{1}{2}\ln(\sec w + \tan w)$



Por otro lado, tenemos que $u = x+1 = \tan w$

$\Rightarrow \sec w = \sqrt{(x+1)^2+1}$

$$\therefore \int \frac{dx}{\sqrt{4x^2+8x+8}} = \frac{1}{2}\ln\left(\sqrt{(x+1)^2+1} + x+1\right) + C$$

⑥ $\displaystyle\int \frac{\ln(\ln(\ln x))\,dx}{x \ln x \ln(\ln x)}$

En 1era. instancia puede parecer muy aparatoso resolver esta $\int$, pero si hacemos $u = \ln(\ln(\ln x))$

$\Rightarrow du = \frac{1}{\ln(\ln x)}\,\frac{1}{\ln x}\,\frac{1}{x}\,dx$

$\Rightarrow$ la $\int$ se transforma en

$$\int \frac{\ln(\ln(\ln x))\,dx}{x\ln x \ln(\ln x)} = \int u\,du = \frac{u^2}{2} = \frac{1}{2}(\ln(\ln(\ln x)))^2 + C$$

⑦ $\displaystyle\int 2^x 2^{2^x}dx$

sabemos que $2^{2^x} = e^{2^x \ln 2} = e^{e^{x\ln 2}\ln 2} = u \Rightarrow du = e^{e^{x\ln 2}\ln 2}(\ln 2)\,e^{x\ln 2}(\ln 2)\,dx$

$\Rightarrow du = 2^x 2^{2^x}(\ln 2)^2 dx \Rightarrow du = 2^x(\ln 2)^2 u\,dx \Rightarrow 2^x dx = \frac{du}{u(\ln 2)^2}$

$\Rightarrow \int 2^x 2^{2^x} dx = \frac{1}{(\ln 2)^2}\int \frac{u\,du}{u} = \frac{u}{(\ln 2)^2}$

$$\therefore \int 2^x 2^{2^x}dx = \frac{2^{2^x}}{(\ln 2)^2} + C$$

⑧ Resolver $\int_0^4 x^3 dx$ con sumas de Riemann.

Utilizando $\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(a+\Delta_x i)\Delta_x$ con $\Delta_x = \frac{b-a}{n}$

$$\Rightarrow \Delta_x = \frac{4-0}{n} = \frac{4}{n}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{4i}{n}\right)\frac{4}{n} = \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{4i}{n}\right)^3 \frac{4}{n} = \lim_{n\to\infty} \left(\frac{4}{n}\right)^4 \sum_{i=1}^{n} i^3 = \lim_{n\to\infty} \frac{4^4}{n^4}\frac{n^2(n+1)^2}{4}$$

$$= 64 \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 = 64 \lim_{n\to\infty} \left(\cancel{1+\frac{1}{n}}\right)^2 = 64$$

$$\therefore \int_0^4 x^3 dx = 64$$

⑨ $\int_{-\infty}^{\infty} e^{-x^2} dx$

Para resolver esto, tendremos que resolver no la $\int$, sino su cuadrado, i.e.

$$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-a^2}da\right)\left(\int_{-\infty}^{\infty} e^{-b^2}db\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(a^2+b^2)}da\,db$$

haciendo cambio a coordenadas polares $a = r\cos\theta$, $b = r\sin\theta$ y recordando que el Jacobiano asociado a dichas coordenadas es r $\Rightarrow$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(a^2+b^2)}da\,db = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2} dr\,d\theta = \int_0^{2\pi} d\theta \int_0^{\infty} r e^{-r^2}dr = 2\pi \int_0^{\infty} r e^{-r^2} dr \cdots \otimes$$

donde los límites de integración barren todo $\mathbb{R}^2$ en el plano polar

haciendo $u = r^2 \Rightarrow du = 2r\,dr$

$$\Rightarrow \otimes = \frac{\cancel{2}\pi}{\cancel{2}}\int_0^{\infty} e^{-u}du = -\pi e^{-u} = -\pi e^{-r^2}\Big|_0^{\infty} = -\pi(e^{-\infty} - e^{0}) = -\pi(0-1) = \pi$$

Como $\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \pi$ $\quad\therefore \int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

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(10) $\displaystyle\int \frac{(x+1)\,dx}{x^2+5x+6} = \int \frac{(x+1)\,dx}{(x+3)(x+2)}$

Procedemos por fracciones parciales:

$$\frac{x+1}{(x+3)(x+2)} = \frac{A}{x+3} + \frac{B}{x+2} = \frac{A(x+2)+B(x+3)}{(x+3)(x+2)}$$

$\Rightarrow x+1 = A(x+2) + B(x+3)$

si $x=-2 \Rightarrow -1 = B(1) \Rightarrow B=-1$

si $x=-3 \Rightarrow -2 = A(-1) \Rightarrow A=2$

$$\Rightarrow \frac{x+1}{(x+3)(x+2)} = \frac{2}{x+3} - \frac{1}{x+2}$$

$$\therefore \int \frac{(x+1)\,dx}{(x+3)(x+2)} = \int \left(\frac{2}{x+3} - \frac{1}{x+2}\right)dx = 2\int \frac{dx}{x+3} - \int \frac{dx}{x+2}$$

$$= 2\ln(x+3) - \ln(x+2)$$

$$\therefore \int \frac{(x+1)\,dx}{x^2+5x+6} = \ln\left(\frac{(x+3)^2}{x+2}\right) + C$$