11/09/2014

① Demostrar que $\int_0^\infty e^{-xt}\frac{\operatorname{sen} t}{t}dt$ converge uniformemente en $\{x \geq 0\}$.

Sol.

Tenemos que $\int_0^\infty e^{-xt}\frac{\operatorname{sen} t}{t}dt$ converge uniformemente si dado $\varepsilon > 0$ $\exists \eta(\varepsilon) \ni \forall \eta, \beta > \eta(\varepsilon)$ se tiene que

$$\left|\int_\eta^\beta e^{-xt}\frac{\operatorname{sen} t}{t}dt\right| < \varepsilon$$

$$\Rightarrow \int_\eta^\beta \frac{e^{-xt}}{t}\operatorname{sen} t\, dt = -\operatorname{Cos} t\,\frac{e^{-xt}}{t}\Big|_\eta^\beta + \int_\eta^\beta \frac{(1+xt)e^{-xt}(-\operatorname{Cos} t)}{t^2}dt$$

$u = \frac{e^{-xt}}{t}$ $\quad dv = \operatorname{sen} t\, dt$

$du = \frac{e^{-xt}(-xt-1)}{t^2}$ $\quad v = -\operatorname{Cos} t$

$$= \frac{e^{-x\eta}}{\eta}\operatorname{Cos}\eta + \int_\eta^\beta \frac{(1+xt)e^{-xt}(-\operatorname{Cos} t)}{t^2}dt \quad \dots ①$$

Como $\frac{e^{-x\eta}}{\eta}\operatorname{Cos}\eta \leq \frac{1}{\eta}$ y $|(1+xt)e^{-xt}(-\operatorname{Cos} t)| \leq e^{xt}e^{-xt}|-\operatorname{Cos} t| \leq 1$

$\Rightarrow$

$$① \leq \frac{1}{\eta} + \int_\eta^\beta \frac{dt}{t^2} = \frac{1}{\eta} - \frac{1}{t}\Big|_\eta^\beta = \frac{2}{\eta} \quad \therefore \frac{2}{\eta} < \varepsilon \iff \frac{2}{\varepsilon} < \eta = \eta(\varepsilon)$$

$$\therefore \left|\int_\eta^\beta e^{-xt}\frac{\operatorname{sen} t}{t}dt\right| < \varepsilon \text{ si } \eta, \beta > \frac{2}{\varepsilon} \quad \blacktriangle$$

② Evaluar la integral $\int_0^\infty e^{-t^2}\operatorname{Cos}(xt)\,dt$

Por un lado, $|e^{-t^2}\operatorname{Cos}(xt)| \leq e^{-t^2}$ y como $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$

$\Rightarrow \int_0^\infty e^{-t^2}\operatorname{Cos}(xt)\,dt$ converge.

Ahora, derivando bajo el signo de la integral tenemos

$$f(x)=\int_0^\infty e^{-t^2}\cos(xt)\,dt \Rightarrow f'(x)=-\int_0^\infty t e^{-t^2}\operatorname{sen}(xt)\,dt$$

$$u=\operatorname{sen}(xt) \qquad dv=te^{-t^2}$$
$$du=x\cos(xt)dt \qquad v=-\frac{e^{-t^2}}{2}$$

$$=-\left(-\frac{e^{-t^2}}{2}\operatorname{sen}(xt)\Big|_0^\infty+\frac{x}{2}\int_0^\infty e^{-t^2}\cos(xt)\,dt\right)=-\frac{x}{2}f(x)$$

$$\therefore f'(x)=-\frac{x}{2}f(x) \Rightarrow \frac{f'(x)}{f(x)}=-\frac{x}{2} \Rightarrow \ln(f(x))=-\frac{x^2}{4}+C_0$$

$$\Rightarrow f(x)=e^{-\frac{x^2}{4}+C_0}=e^{C_0}e^{-x^2/4}=ce^{-x^2/4}$$

$$\therefore f(x)=ce^{-x^2/4}$$

Como $f(0)=\int_0^\infty e^{-t^2}\cos(0t)\,dt=\int_0^\infty e^{-t^2}dt=\frac{\sqrt{\pi}}{2}$ y

$$f(0)=ce^0=c \qquad \Rightarrow c=\frac{\sqrt{\pi}}{2}$$

$$\therefore f(x)=\frac{\sqrt{\pi}}{2}e^{-\frac{x^2}{4}}$$

③ Calcular el valor de $I=\int_0^\infty \frac{\arctan x}{x(1+x^2)}dx$

Para calcular I usamos la integral paramétrica

$J(a)=\int_0^\infty \frac{\arctan(ax)}{x(1+x^2)}dx$ y la derivamos usando la regla de Leibniz.

$$\Rightarrow J'(a)=\int_0^\infty \frac{\cancel{x}\,dx}{(1+a^2x^2)\cancel{x}(1+x^2)}=\int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)} \quad \cdots ①$$

Usando fracciones parciales tenemos que

$$\frac{A}{1+x^2}+\frac{B}{1+a^2x^2}=\frac{A(1+a^2x^2)+B(1+x^2)}{(1+x^2)(1+a^2x^2)} \Rightarrow \begin{array}{l} A+B=1 \rightarrow B=1-A \\ a^2A+B=0 \rightarrow a^2A+1-A=0 \end{array}$$

$$\Rightarrow A=\frac{1}{1-a^2} \quad y \quad B=1-A=\frac{-a^2}{1-a^2}$$

11/09/2014

$\therefore$ ① $= \dfrac{1}{1-a^2}\displaystyle\int_0^\infty \left(\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right)dx = \frac{1}{1-a^2}\left(\frac{\pi}{2} - \frac{a\pi}{2}\right) = \frac{\pi}{2(1+a)}$

$\arctan x \Big|_0^\infty = \dfrac{\pi}{2}$

$x = \frac{1}{a}\tan\theta$

$dx = \frac{1}{a}\sec^2\theta\, d\theta$

$\Rightarrow \displaystyle\int_0^\infty \frac{a^2}{1+a^2x^2}dx = \int_0^\infty \frac{a^2 \frac{1}{a}\sec^2\theta\, d\theta}{1+\tan^2\theta} = a\theta$

$= a\arctan(ax)\Big|_0^\infty = \dfrac{a\pi}{2}$

$\therefore\ J'(a) = \dfrac{\pi}{2(1+a)} \Rightarrow J(a) = \dfrac{\pi}{2}\ln(1+a) + c$

Ahora, $J(0) = \displaystyle\int_0^\infty \frac{\arctan(0x)}{x(1+x^2)}dx = 0$ y $J(0) = \frac{\pi}{2}\ln(1+0) + c = c$

$\Rightarrow c = 0$

$\therefore\ J(a) = \dfrac{\pi}{2}\ln(1+a)$

$\therefore\ I = J(1) = \dfrac{\pi}{2}\ln 2$

④ Calcular el valor de $I = \displaystyle\int_0^\infty \frac{1-e^{-x^2}}{x^2}dx$

Usando la integral paramétrica $J(a) = \displaystyle\int_0^\infty \frac{1-e^{-ax^2}}{x^2}dx$

$\Rightarrow J'(a) = \displaystyle\int_0^\infty \frac{\cancel{x^2}e^{-ax^2}}{\cancel{x^2}}dx = \int_0^\infty e^{-ax^2}dx = \frac{1}{\sqrt{a}}\int_0^\infty e^{-u^2}du = \frac{1}{\sqrt{a}}\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{a}}$

$u^2 = ax^2$

$\cancel{2}u\,du = \cancel{2}ax\,dx$

$\Rightarrow dx = \dfrac{u\,du}{ax} = \dfrac{\cancel{u}\,du}{a\frac{\cancel{u}}{\sqrt{a}}} = \dfrac{du}{\sqrt{a}}$

$\therefore\ J'(a) = \dfrac{1}{2}\sqrt{\dfrac{\pi}{a}} \Rightarrow J(a) = \dfrac{\sqrt{\pi}}{2}2\sqrt{a} + c = \sqrt{a\pi} + c$

$\therefore\ J(a) = \sqrt{\pi a} + c$

Ahora, $J(0) = \displaystyle\int_0^\infty \frac{1-e^{0x^2}}{x^2}dx = 0$ y $J(0) = \sqrt{\pi(0)} + c = c$

$\Rightarrow c = 0$ $\quad\therefore\ J(a) = \sqrt{\pi a}$

$\therefore\ I = J(1) = \sqrt{\pi}$