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Integrales impropias sobre regiones no acotadas

1) Calcular $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy$

Haciendo cambio a polares tenemos que

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^{\infty} r e^{-r^2}\,dr\,d\theta = \lim_{n\to\infty} 2\pi\int_0^{n} r e^{-r^2}\,dr = -\pi \lim_{n\to\infty}\int_0^{n} e^{u}\,du$$

$u=-r^2$
$du=-2r\,dr$

$$= -\pi \lim_{n\to\infty}(\cancel{e^{-n^2}}^{\,0} - 1) = \pi$$

2) Evaluar $\int_0^{\infty}\frac{\operatorname{sen} x}{x}\,dx$

Para calcular esta integral primero vamos a calcular lo siguiente:

a) $\int_0^{\infty} e^{-xy}\,dy$ cuando $x>0$

$$\Rightarrow \int_0^{\infty} e^{-xy}\,dy = -\frac{1}{x}\lim_{n\to\infty}\int_0^{n} e^{u}\,du = -\frac{1}{x}\lim_{n\to\infty}(e^{-nx}-1) = \frac{1}{x} \quad \text{si } x>0$$

$u=-xy$
$du=-x\,dy$

b) $\int_0^{\infty} e^{-xy}\operatorname{sen} x\,dx$ cuando $y>0$

$$\Rightarrow \int_0^{\infty} e^{-xy}\operatorname{sen} x\,dx = \lim_{n\to\infty}\left[-\frac{\operatorname{sen} x}{y}e^{-xy}\Big|_0^n + \int_0^n \frac{\cos x\, e^{-xy}}{y}\,dx\right]$$

$u=\operatorname{sen} x$ $\quad dv = e^{-xy}dx$
$du=\cos x\,dx$ $\quad v=-\frac{e^{-xy}}{y}$

$u=\cos x$ $\quad dv=e^{-xy}dx$
$du=-\operatorname{sen} x\,dx$ $\quad v=-\frac{e^{-xy}}{y}$

$$= \lim_{n\to\infty}\left[-\cancel{\frac{\operatorname{sen} n\, e^{-ny}}{y}}^{\,0} + \cancel{\frac{\operatorname{sen}(0)e^{-0}}{y}}^{\,0} + \frac{1}{y}\left(-\frac{\cos x\, e^{-xy}}{y}\Big|_0^n - \int_0^n \frac{\operatorname{sen} x\, e^{-xy}}{y}\,dx\right)\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{y^2}\cos(0)\,e^{-0} - \frac{1}{y^2}\cancel{\cos(n)\,e^{-ny}}^{\,0} - \frac{1}{y^2}\int_0^n \operatorname{sen}x\, e^{-xy}dx \right]$$

$$\Rightarrow \left(1+\frac{1}{y^2}\right)\lim_{n\to\infty}\int_0^n \operatorname{Sen}x\, e^{-xy}dx = \frac{1}{y^2}$$

$$\therefore \int_0^\infty \operatorname{sen}x\, e^{-xy}dx = \frac{1}{1+y^2} \quad \text{si } y>0$$

∴ retomando la 1era. integral

$$\int_0^\infty \frac{\operatorname{Sen}x}{x}dx \underset{\text{por a)}}{=} \int_0^\infty\int_0^\infty \operatorname{sen}x\, e^{-xy}dx\,dy \underset{\text{por b)}}{=} \int_0^\infty \frac{dy}{1+y^2} = \lim_{n\to\infty}\left[\arctan(n)-\arctan(0)\right]$$

$$= \frac{\pi}{2}$$

3) Calcular $\Gamma(\frac{1}{2})$

Recordando que $\Gamma(n) = \int_0^\infty e^{-t}t^{n-1}dt$

$$\Rightarrow \Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty e^{-t}t^{-1/2}dt = \int_0^\infty e^{-x^2}x^{-1}2x\,dx = \int_0^\infty 2e^{-x^2}dx = \int_{-\infty}^{\infty} e^{-x^2}dx$$

$t=x^2$
$dt=2x\,dx$

por ser una función par

La última integral ya se había visto en clase y es igual a $\sqrt{\pi}$.

$$\therefore \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

4) Calcular $\int_0^\infty x^{1/2}e^{-x^3}dx$

Sea $u=x^3 \rightarrow x=u^{1/3}$
$du=3x^2dx \rightarrow dx=\frac{1}{3}u^{-2/3}du$

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$$\therefore \int_0^\infty x^{1/2} e^{-x^3} dx = \int_0^\infty u^{1/6} e^{-u} \frac{1}{3} u^{-2/3} du = \frac{1}{3} \underbrace{\int_0^\infty e^{-u} u^{-1/2} du}_{\Gamma(\frac{1}{2})} = \frac{\sqrt{\pi}}{3}$$

5) Demostrar que $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt = \int_0^\infty u^{2x-2} e^{-u^2} 2u\, du = 2\int_0^\infty u^{2x-1} e^{-u^2} du \quad \ldots \text{①}$$

$t = u^2$
$dt = 2u\,du$

$$\Gamma(x+y) = \int_0^\infty t^{x+y-1} e^{-t} dt = 2\int_0^\infty r^{2(x+y)-1} e^{-r^2} dr \quad \ldots \text{②}$$

$t = r^2$
$dt = 2r\,dr$

La función beta de 2 variables definida en el 1er. cuadrante del plano xy es

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} dt = -\int_1^0 (1-u)^{x-1} u^{y-1} dt = \int_0^1 u^{y-1}(1-u)^{x-1} du$$

$u = 1-t \rightarrow t = 1-u$
$du = -dt$

En donde observamos que si $x<1$ y/o $y<1 \Rightarrow$ la integral es impropia. Si sustituimos $t = \operatorname{sen}^2\theta \Rightarrow dt = 2\operatorname{sen}\theta\cos\theta\, d\theta$

$$\Rightarrow \int_0^1 t^{x-1}(1-t)^{y-1} dt = \int_0^{\frac{\pi}{2}} (\operatorname{sen}^2\theta)^{x-1}(1-\operatorname{sen}^2\theta)^{y-1} 2\operatorname{sen}\theta\cos\theta\, d\theta$$

$$= 2\int_0^{\frac{\pi}{2}} (\operatorname{sen}\theta)^{2x-1}(\cos\theta)^{2y-1} d\theta \quad \ldots \text{③}$$

Ahora, $\Gamma(x)\Gamma(y) \underset{\text{de ①}}{=} \left(2\int_0^\infty u^{2x-1} e^{-u^2} du\right)\left(2\int_0^\infty v^{2y-1} e^{-v^2} dv\right)$

$$= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\, u^{2x-1} v^{2y-1}\, du\, dv \quad \text{haciendo cambio a polares}$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty r e^{-r^2} (r\cos\theta)^{2x-1} (r\,\text{sen}\,\theta)^{2y-1}\, dr\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} (\cos\theta)^{2x-1} (\text{sen}\,\theta)^{2y-1}\, r^{2(x+y)-1}\, dr\, d\theta$$

$$= \underbrace{2\int_0^{\frac{\pi}{2}} (\cos\theta)^{2x-1} (\text{sen}\,\theta)^{2y-1}\, d\theta}_{B(x,y)\ (\text{por } ③)}\ \underbrace{2\int_0^\infty e^{-r^2}\, r^{2(x+y)-1}\, dr}_{\Gamma(x+y)\ (\text{por } ②)}$$

$$\therefore\ \Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)$$

$$\therefore\ B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

6) Calcular $\Gamma(1)$

$$\Gamma(1) = \int_0^\infty e^{-t} t^{1-1}\, dt = \int_0^\infty e^{-t}\, dt = \lim_{n\to\infty} -[e^{-n} - 1] = 1$$

7) Demostrar que $\Gamma(x+1) = x\Gamma(x)$

$$\Gamma(x+1) = \int_0^\infty e^{-t} t^{x+1-1}\, dt = \int_0^\infty e^{-t} t^x\, dt = \lim_{n\to\infty}\left[\cancel{-t^x e^{-t}}\Big|_0^n + x\underbrace{\int_0^n e^{-t} t^{x-1}\, dt}_{\Gamma(x)} \right]$$

$u = t^x \qquad dv = e^{-t}\, dt$

$du = x t^{x-1}\, dt \qquad v = -e^{-t}$

$$\therefore\ \Gamma(x+1) = x\Gamma(x)$$

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8) Demostrar que $\Gamma(n) = (n-1)! \quad \forall n \in \mathbb{N}$

Por inducción.

$n = 1$

$\Gamma(1) = (1-1)! = 0! = 1$ por 6)

Suponemos cierto para $n \Rightarrow \Gamma(n) = (n-1)!$

$\Rightarrow$ para $\underbrace{\Gamma(n+1) = n\Gamma(n)}_{\text{por } 7)} \underbrace{=}_{\text{por H. I.}} n(n-1)! = n!$

$\therefore \; \Gamma(n) = (n-1)!$

9) Calcular $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{sen}^5\theta \cos^7\theta \, d\theta = \frac{1}{2} B(4,3) = \frac{1}{2} \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}$

$= \dfrac{1}{2} \dfrac{2!\,3!}{6!} = \dfrac{1}{120}$