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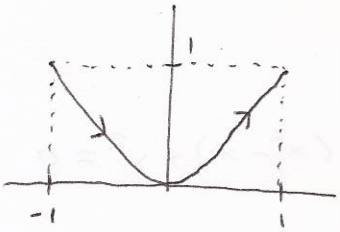
Integral de línea, campos conservativos y función potencial

① Calcule la integral del campo vectorial $F(x,y) = (x^2 - 2xy)\hat{i} + (y^2 - 2xy)\hat{j}$ a lo largo de la parábola $y = x^2$ desde $(-1,1)$ a $(1,1)$.

Sol.



Sea $x = t \Rightarrow y = t^2$

$\therefore\ \sigma(t) = (t, t^2)$ donde $t \in [-1, 1]$

en donde se puede ver que $\sigma(-1) = (-1, 1)$, $\sigma(0) = (0,0)$ y $\sigma(1) = (1,1)$

$\Rightarrow \sigma'(t) = (1, 2t)$ y $F(\sigma(t)) = (t^2 - 2t^3,\ t^4 - 2t^3)$

$$\Rightarrow \int_{-1}^{1} F(\sigma(t))\cdot\sigma'(t)\,dt = \int_{-1}^{1} (t^2 - 2t^3 + 2t^5 - 4t^4)\,dt$$

$$= \left(\frac{t^3}{3} - \frac{t^4}{2} + \frac{t^6}{3} - \frac{4}{5}t^5\right)\Big|_{-1}^{1} = \frac{1}{3} + \frac{1}{3} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{3}} - \frac{4}{5} - \frac{4}{5} = -\frac{14}{15}$$

Nota:

Otra manera común de escribir las integrales de línea es

$$\int_\sigma F\cdot ds = \int_\sigma F_1\,dx + F_2\,dy + F_3\,dz$$

donde F_1, F_2, F_3 son las componentes del campo vectorial F. A la expresión $F_1\,dx + F_2\,dy + F_3\,dz$ la llamamos forma diferencial. Por definición, la integral de una forma diferencial es:

$$\int_\sigma F_1\,dx + F_2\,dy + F_3\,dz = \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right)dt = \int_\sigma F\cdot ds$$

② Calcule la integral de línea $\int_\sigma (x+2)\,dx + 3z\,dy + y^2\,dz$ siendo σ una parametrización de la curva intersección de las superficies:

$$x^2+y^2+z^2=1 \quad , \quad z = x-1$$

Sol.

Sustituyendo $z=x-1$ en la ec. de la esfera

$$x^2+y^2+x^2-2x+1=1 \Rightarrow 2x^2+y^2-2x=0 \Rightarrow 2(x^2-x)+y^2=0$$

$$\Rightarrow 2\left(x^2-x+\tfrac14\right)+y^2=\tfrac12 \Rightarrow \frac{2\left(x-\frac12\right)^2}{\frac12}+\frac{y^2}{\frac12}=1$$

$$\Rightarrow \frac{\left(x-\frac12\right)^2}{\frac14}+\frac{y^2}{\frac12}=1$$

$\Rightarrow$ Haciendo $\dfrac{x-\frac12}{\frac12}=\cos t \quad , \quad \dfrac{y}{\frac{1}{\sqrt2}}=\text{sen}\,t \quad , \quad z=x-1$

$$\Rightarrow x=\frac12+\frac12\cos t \quad , \quad y=\frac{1}{\sqrt2}\,\text{sen}\,t \quad , \quad z=-\frac12+\frac12\cos t \qquad t\in[0,2\pi]$$

$$\therefore \sigma(t)=\frac12\left(1+\cos t,\ \sqrt2\,\text{sen}\,t,\ \cos t-1\right) \qquad t\in[0,2\pi]$$

$$\Rightarrow \sigma'(t)=\frac12\left(-\text{sen}\,t,\ \sqrt2\cos t,\ -\text{sen}\,t\right)$$

$$\therefore \int_\sigma (x+2)\,dx+3z\,dy+y^2\,dz =$$

$$\int_0^{2\pi}\left(\left(\frac12(1+\cos t)+2\right)\frac12(-\text{sen}\,t)+\frac32(\cos t-1)\frac{\sqrt2}{2}(\cos t)+\left(\frac{\sqrt2}{2}\,\text{sen}\,t\right)^2\left(-\frac12\,\text{sen}\,t\right)\right)$$

$$=\frac14\int_0^{2\pi}\left(-5\,\text{sen}\,t-\text{sen}\,t\cos t+3\sqrt2\cos^2 t-3\sqrt2\cos t-\text{sen}\,t\,(1-\cos^2 t)\right)dt$$

$$=\frac14\int_0^{2\pi}\left(-6\,\text{sen}\,t-\text{sen}\,t\cos t+3\sqrt2\left(\frac{1+\cos 2t}{2}\right)-3\sqrt2\cos t+\text{sen}\,t\cos^2 t\right)dt$$

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$$= \frac{1}{4}\left(6\cos t + \frac{\cos^2 t}{2} + \frac{3\sqrt{21}}{2}\left(t + \frac{1}{2}\operatorname{sen}2t\right) - 3\sqrt{21}\operatorname{sen}t - \frac{\cos^3 t}{3}\right)\Big|_0^{2\pi}$$

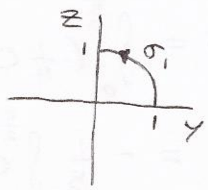
$$= \frac{1}{4}\left(\frac{3\sqrt{21}}{2}(2\pi)\right) = \frac{3\sqrt{21}}{4}\pi$$

③ Calcule la integral $\int_\sigma x^2y\,dx + 2y\,dy + x\,dz$ a lo largo del camino cerrado σ, limitado por los arcos $\sigma_1, \sigma_2, \sigma_3$ dados por

$$\sigma_1 \begin{cases} x^2+y^2+z^2=1 \\ x=0 \\ y\geq 0,\ z\geq 0 \end{cases}, \quad \sigma_2 \begin{cases} 2x+z=1 \\ y=0 \\ x\geq 0,\ z\geq 0 \end{cases}, \quad \sigma_3 \begin{cases} 4x^2+y^2=1 \\ z=0 \\ x\geq 0,\ y\geq 0 \end{cases}$$

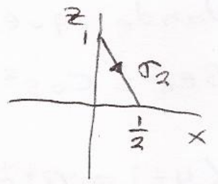
Sol.

De σ_1 obtenemos que para $x=0 \Rightarrow y^2+z^2=1$



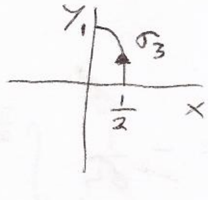
$\Rightarrow x=0,\ y=\cos t,\ z=\operatorname{sen}t \quad t\in[0,\frac{\pi}{2}]$

$\Rightarrow \sigma_1(t)=(0,\cos t,\operatorname{sen}t)$, $\sigma_1(0)=(0,1,0)$
$\sigma_1(\frac{\pi}{2})=(0,0,1)$



Para σ_2 sea $x=t \Rightarrow z=1-2t \quad t\in[0,\frac{1}{2}]$

$\Rightarrow \sigma_2(t)=(t,0,1-2t)$, $\sigma_2(0)=(0,0,1)$
$\sigma_2(\frac{1}{2})=(\frac{1}{2},0,0)$

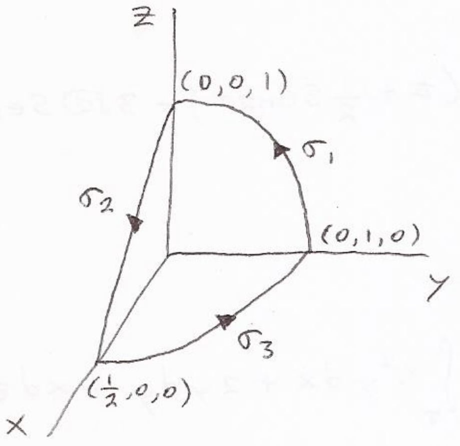


Para σ_3 sea $x=\frac{1}{2}\cos t$, $y=\operatorname{sen}t \quad t\in[0,\frac{\pi}{2}]$

$\Rightarrow \sigma_3(t)=(\frac{1}{2}\cos t,\operatorname{sen}t,0)$, $\sigma_3(0)=(\frac{1}{2},0,0)$
$\sigma_3(\frac{\pi}{2})=(0,1,0)$

$\Rightarrow \sigma_1(0)=\sigma_3(\frac{\pi}{2})$, $\sigma_1(\frac{\pi}{2})=\sigma_2(0)$ y $\sigma_2(\frac{1}{2})=\sigma_3(0)$,

por lo cual es un ciclo cerrado, i,e



$$\sigma = \sigma_1 \cup \sigma_2 \cup \sigma_3$$

$$\Rightarrow \int_\sigma = \int_{\sigma_1} + \int_{\sigma_2} + \int_{\sigma_3}$$

$$\int_{\sigma_1} F \cdot ds = \int_0^{\frac{\pi}{2}} (2\operatorname{Cos} t(-\operatorname{Sen} t))\,dt = \operatorname{Cos}^2 t \Big|_0^{\frac{\pi}{2}} = -1$$

$$\int_{\sigma_2} F \cdot ds = \int_0^{\frac{1}{2}} (t(-2))\,dt = -t^2 \Big|_0^{\frac{1}{2}} = -\frac{1}{4}$$

$$\int_{\sigma_3} F \cdot ds = \int_0^{\frac{\pi}{2}} \left(\frac{1}{4}\operatorname{Cos}^2 t \operatorname{Sen} t\left(-\frac{1}{2}\operatorname{Sen} t\right) + 2\operatorname{Sen} t(\operatorname{Cos} t)\right)dt$$

$$= -\frac{1}{8}\int_0^{\frac{\pi}{2}} \operatorname{Cos}^2 t \operatorname{Sen}^2 t\,dt + \operatorname{Sen}^2 t \Big|_0^{\frac{\pi}{2}} = -\frac{1}{8}\int_0^{\frac{\pi}{2}} \operatorname{Cos}^2 t \operatorname{Sen}^2 t\,dt + 1$$

// Recordando que $\operatorname{Sen}(2t) = \operatorname{Sen} t \operatorname{Cos} t + \operatorname{Sen} t \operatorname{Cos} t = 2\operatorname{Sen} t \operatorname{Cos} t$

$$\Rightarrow \operatorname{Sen}^2 t \operatorname{Cos}^2 t = \frac{\operatorname{Sen}^2(2t)}{4} \quad \text{y}$$

$$\operatorname{Cos}(4t) = \operatorname{Cos}^2(2t) - \operatorname{Sen}^2(2t) = 1 - 2\operatorname{Sen}^2(2t) \Rightarrow \operatorname{Sen}^2(2t) = \frac{1-\operatorname{Cos}(4t)}{2}$$

$$\Rightarrow \operatorname{Sen}^2 t \operatorname{Cos}^2 t = \frac{1-\operatorname{Cos}(4t)}{8} \quad //$$

$$\Rightarrow \quad = -\frac{1}{64}\int_0^{\frac{\pi}{2}} (1-\operatorname{Cos}(4t))\,dt + 1 = -\frac{1}{64}\left(t - \frac{1}{4}\operatorname{Sen}(4t)\right)\Big|_0^{\frac{\pi}{2}} + 1$$

$$= -\frac{\pi}{128} + 1$$

$$\therefore \int_\sigma x^2 y\,dx + 2y\,dy + x\,dz = -1 - \frac{1}{4} - \frac{\pi}{128} + 1 = -\frac{1}{4} - \frac{\pi}{128}$$

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(4) Pruebe que la integral $\int_\sigma (6xy^2-y^3)\,dx+(6x^2y-3xy^2)\,dy$ es independiente del camino que une los puntos $(1,2)$ con $(3,4)$ y calcule el valor de la integral

a) parametrizando el segmento

b) utilizando la función potencial del integrando.

Sol.

La integral será independiente del camino si el campo

$$F(x,y)=(6xy^2-y^3,\ 6x^2y-3xy^2)$$

es conservativo. Como $F:\mathbb{R}^2\to\mathbb{R}^2$ bastará con comprobar que se cumple $\partial_x F_2=\partial_y F_1\ \forall (x,y)\in\mathbb{R}^2$. En efecto,

$$\partial_x F_2 = 12xy-3y^2 \quad y \quad \partial_y F_1 = 12xy-3y^2$$

$\therefore$ F es conservativo y la integral es independiente del camino.

a) Sea $\sigma(t)=(1,2)+t((3,4)-(1,2))=(1,2)+t(2,2)$

$\Rightarrow \sigma(t)=(1+2t,2+2t)\quad t\in[0,1]$ el segmento que une ambos puntos.

$$\Rightarrow \int_0^1 (6(1+2t)(2+2t)^2-(2+2t)^3+6(1+2t)^2(2+2t)-3(1+2t)(2+2t)^2)(2)\,dt$$

$$=2\int_0^1((6+12t)(4+8t+4t^2)-(8+24t+24t^2+8t^3)+(1+4t+4t^2)(12+12t)$$
$$-(3+6t)(4+8t+4t^2))\,dt$$

$$=2\int_0^1(24+48t+24t^2+48t+96t^2+48t^3-8-24t-24t^2-8t^3+12+12t+48t$$
$$+48t^2+48t^2+48t^3-12-24t-12t^2-24t-48t^2-24t^3)\,dt$$

$$=2\int_0^1(16+84t+132t^2+64t^3)\,dt=2(16t+42t^2+44t^3+16t^4)\Big|_0^1$$

$$=2(16+42+44+16)=2(118)=236$$

b) Ahora, para calcular f $(\nabla f = F)$ tenemos que

$$\partial_x f(x,y) = 6xy^2 - y^3 \quad \dots 1)$$
$$\partial_y f(x,y) = 6x^2y - 3xy^2 \quad \dots 2)$$

Integrando a 1) $\Rightarrow f(x,y) = 3x^2y^2 - xy^3 + h(y)$

Derivando el último resultado respecto a y y comparando con 2) tenemos

$$\partial_y f(x,y) = 6x^2y - 3xy^2 + h'(y) \quad \Rightarrow h'(y) = 0 \quad \Rightarrow h(y) = K = cte.$$

$$\therefore f(x,y) = 3x^2y^2 - xy^3 + K$$

$$\Rightarrow \int_\sigma F = \int_\sigma \nabla f = f(\sigma(1)) - f(\sigma(0)) = f(3,4) - f(1,2) = 3(3^2 \cdot 4^2) - 3 \cdot 4^3 + K$$
$$- 3(1^2 \cdot 2^2) + 1 \cdot 2^3 - K$$
$$= 240 - 4 = 236$$

⑤ Halle el valor del parámetro a para que el campo vectorial

$$F(x,y,z) = (z, az^2 + 1, 10zy + x)$$

sea conservativo, y para dicho valor a obtenga una función potencial de F.

Sol.

Para que F sea conservativo se necesita que $\nabla \times F = 0 \Rightarrow$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ z & az^2+1 & 10zy+x \end{vmatrix} = \hat{i}(10z - 2az) - \hat{j}(1-1) + \hat{k}(0-0) = 0$$

$$\Rightarrow a = 5$$

$\therefore$ Si $a = 5$ F es conservativo, y $\exists f \ni \nabla f = F$

$$\Rightarrow \partial_x f = z \quad \dots 1)$$
$$\partial_y f = 5z^2 + 1 \quad \dots 2)$$
$$\partial_z f = 10zy + x \quad \dots 3)$$

$\Rightarrow$ de 1) $f = zx + h(y,z) \Rightarrow \partial_z f = x + h'(y,z)$ comparando con 3)

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$h'(y,z) = 10zy \quad \Rightarrow h(y,z) = 5z^2y + k(y)$

$\Rightarrow f(x,y,z) = zx + 5z^2y + k(y)$

$\Rightarrow \partial_y f = 5z^2 + k'(y)$ y comparando con 2) $\Rightarrow k'(y) = 1$

$\Rightarrow k(y) = y + C$ con C=cte.

$\therefore$ $f(x,y,z) = zx + 5z^2y + y + C$