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Propiedades de gradiente, rotacional y divergencia

Sea $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ un campo vectorial diferenciable y sea $f(x,y,z)$ una función escalar, también diferenciable.

Usemos $\frac{\partial}{\partial x_i} \equiv \partial_{x_i}$ con $x_1 = x$, $x_2 = y$ y $x_3 = z$, y

$\frac{\partial^2}{\partial x_i \partial x_j} \equiv \partial_{x_i x_j}$ para $x_i \neq x_j$ y $\partial^2_{x_i}$ para $x_i = x_j$.

$\nabla = (\partial_x, \partial_y, \partial_z)$

•) $\boxed{\operatorname{div}(\operatorname{rot} F) = \nabla \cdot (\nabla \times F) = 0}$

$$\nabla \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix} = (\partial_y R - \partial_z Q, \partial_z P - \partial_x R, \partial_x Q - \partial_y P) = \star$$

$$\Rightarrow \nabla \cdot (\nabla \times F) = (\partial_x, \partial_y, \partial_z) \cdot \star = \partial^2_{xy} R - \partial^2_{xz} Q + \partial^2_{yz} P - \partial^2_{yx} R + \partial^2_{zx} Q - \partial^2_{zy} P = 0$$

parciales cruzadas son iguales.

•) $\boxed{\nabla \times \nabla f = \bar{0}}$

$\nabla f = (\partial_x f, \partial_y f, \partial_z f)$

$$\Rightarrow \nabla \times \nabla f = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x f & \partial_y f & \partial_z f \end{vmatrix} = (\partial^2_{yz} f - \partial^2_{zy} f, \partial^2_{zx} f - \partial^2_{xz} f, \partial^2_{xy} f - \partial^2_{yx} f) = (0,0,0) = \bar{0}$$

•) $\boxed{\nabla \cdot (fF) = f \nabla \cdot F + F \cdot \nabla f}$

Por un lado

$$\nabla \cdot (fF) = (\partial_x, \partial_y, \partial_z) \cdot (fP, fQ, fR) = f\partial_x P + P\partial_x f + f\partial_y Q + Q\partial_y f + f\partial_z R + R\partial_z f$$

$$= f(\partial_x P + \partial_y Q + \partial_z R) + (P\partial_x f + Q\partial_y f + R\partial_z f)$$

$$= f(\partial_x, \partial_y, \partial_z) \cdot (P,Q,R) + (P,Q,R) \cdot (\partial_x f, \partial_y f, \partial_z f) = f \nabla \cdot F + F \cdot \nabla f$$

•) $\boxed{\nabla\times(\nabla\times F)=\nabla(\nabla\cdot F)-\nabla^2 F}$

Nota: $\nabla^2=\nabla\cdot\nabla=\partial_x^2+\partial_y^2+\partial_z^2$ es el laplaciano.

$$\Rightarrow \nabla\times(\nabla\times F)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ \partial_y R-\partial_z Q & \partial_z P-\partial_x R & \partial_x Q-\partial_y P\end{vmatrix}$$

$$=(\partial^2_{yx}Q-\partial^2_y P-\partial^2_z P+\partial^2_{zx}R,\ \partial^2_{zy}R-\partial^2_z Q-\partial^2_x Q+\partial^2_{xy}P,\ \partial^2_{xz}P-\partial^2_x R-\partial^2_y R+\partial^2_{yz}Q)$$

$$=\#$$

Por otro lado:

$$\nabla(\nabla\cdot F)=(\partial_x,\partial_y,\partial_z)(\partial_x P+\partial_y Q+\partial_z R)$$

$$=(\partial^2_x P+\partial^2_{xy}Q+\partial^2_{xz}R,\ \partial^2_{yx}P+\partial^2_y Q+\partial^2_{yz}R,\ \partial^2_{zx}P+\partial^2_{zy}Q+\partial^2_z R)$$

y $\nabla^2 F=(\partial_x^2+\partial_y^2+\partial_z^2)(P,Q,R)$

$$=(\partial^2_x P+\partial^2_y P+\partial^2_z P,\ \partial^2_x Q+\partial^2_y Q+\partial^2_z Q,\ \partial^2_x R+\partial^2_y R+\partial^2_z R)$$

$$\Rightarrow \nabla(\nabla\cdot F)-\nabla^2 F=(\partial^2_{xy}Q+\partial^2_{xz}R-\partial^2_y P-\partial^2_z P,\ \partial^2_{yx}P+\partial^2_{yz}R-\partial^2_x Q-\partial^2_z Q,\ \partial^2_{zx}P+\partial^2_{zy}Q-\partial^2_x R-\partial^2_y R)$$

En donde se puede ver que lo anterior es igual a #.

$$\therefore \nabla\times(\nabla\times F)=\nabla(\nabla\cdot F)-\nabla^2 F$$

•) $\boxed{\nabla\times(fF)=f\,\nabla\times F+\nabla f\times F}$

Por un lado:

$$\nabla\times(fF)=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ fP & fQ & fR\end{vmatrix}=(f\partial_y R+R\partial_y f-f\partial_z Q-Q\partial_z f,\ f\partial_z P+P\partial_z f-f\partial_x R-R\partial_x f,$$
$$f\partial_x Q+Q\partial_x f-f\partial_y P-P\partial_y f)\ \ldots\ \Delta$$

Y por otro:

$$f\,\nabla\times F=f\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ P & Q & R\end{vmatrix}=(f\partial_y R-f\partial_z Q,\ f\partial_z P-f\partial_x R,\ f\partial_x Q-f\partial_y P)$$

$$y\quad \nabla f\times F=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x f & \partial_y f & \partial_z f\\ P & Q & R\end{vmatrix}=(R\partial_y f-Q\partial_z f,\ P\partial_z f-R\partial_x f,\ Q\partial_x f-P\partial_y f)$$

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En donde se puede ver que sumando los últimos 2 resultados es igual a (A)

$$\therefore \nabla\times(fF) = f\nabla\times F + \nabla f\times F$$

•) $\boxed{\nabla(F\cdot G) = (F\cdot\nabla)G + (G\cdot\nabla)F + F\times(\nabla\times G) + G\times(\nabla\times F)}$

Sea $G=(A,B,C)$ campo vectorial diferenciable.

Por un lado:

$$\nabla(F\cdot G) = (\partial_x,\partial_y,\partial_z)(PA+QB+RC)$$

$$\begin{aligned}= (&P\partial_x A + A\partial_x P + Q\partial_x B + B\partial_x Q + R\partial_x C + C\partial_x R,\\ &P\partial_y A + A\partial_y P + Q\partial_y B + B\partial_y Q + R\partial_y C + C\partial_y R,\\ &P\partial_z A + A\partial_z P + Q\partial_z B + B\partial_z Q + R\partial_z C + C\partial_z R) \quad \dots \text{(I)}\end{aligned}$$

Por otro:

$$(F\cdot\nabla)G = \left[(P,Q,R)\cdot(\partial_x,\partial_y,\partial_z)\right](A,B,C)$$

$$= (P\partial_x + Q\partial_y + R\partial_z)(A,B,C)$$

$$= (P\partial_x A + Q\partial_y A + R\partial_z A,\ P\partial_x B + Q\partial_y B + R\partial_z B,\ P\partial_x C + Q\partial_y C + R\partial_z C)$$

$$(G\cdot\nabla)F = (A\partial_x + B\partial_y + C\partial_z)(P,Q,R)$$

$$= (A\partial_x P + B\partial_y P + C\partial_z P,\ A\partial_x Q + B\partial_y Q + C\partial_z Q,\ A\partial_x R + B\partial_y R + C\partial_z R$$

$$F\times(\nabla\times G) = F\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ A & B & C\end{vmatrix}$$

$$= F\times(\partial_y C - \partial_z B,\ \partial_z A - \partial_x C,\ \partial_x B - \partial_y A)$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ P & Q & R\\ \partial_y C - \partial_z B & \partial_z A - \partial_x C & \partial_x B - \partial_y A\end{vmatrix}$$

$$= (Q\partial_x B - Q\partial_y A - R\partial_z A + R\partial_x C,\ R\partial_y C - R\partial_z B - P\partial_x B + P\partial_y A,$$
$$P\partial_z A - P\partial_x C - Q\partial_y C + Q\partial_z B)$$

y

$$G\times(\nabla\times F) = G\times\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

$$= G\times(\partial_y R - \partial_z Q,\ \partial_z P - \partial_x R,\ \partial_x Q - \partial_y P)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A & B & C \\ \partial_y R - \partial_z Q & \partial_z P - \partial_x R & \partial_x Q - \partial_y P \end{vmatrix}$$

$$= (B\partial_x Q - B\partial_y P - C\partial_z P + C\partial_x R,\ C\partial_y R - C\partial_z Q - A\partial_x Q + A\partial_y P,$$
$$A\partial_z P - A\partial_x R - B\partial_y R + B\partial_z Q)$$

Sumando los últimos 4 resultados:

$$(F\cdot\nabla)G + (G\cdot\nabla)F + F\times(\nabla\times G) + G\times(\nabla\times F) =$$

$$(P\partial_x A + \cancel{Q\partial_y A} + \cancel{R\partial_z A} + A\partial_x P + \cancel{B\partial_y P} + \cancel{C\partial_z P} + Q\partial_x B - \cancel{Q\partial_y A} - \cancel{R\partial_z A} + R\partial_x C$$
$$+ B\partial_x Q - \cancel{B\partial_y P} - \cancel{C\partial_z P} + C\partial_x R,\ \cancel{P\partial_x B} + Q\partial_y B + \cancel{R\partial_z B} + \cancel{A\partial_x Q} + B\partial_y Q + \cancel{C\partial_z Q}$$
$$+ R\partial_y C - \cancel{R\partial_z B} - \cancel{P\partial_x B} + P\partial_y A + C\partial_y R - \cancel{C\partial_z Q} - \cancel{A\partial_x Q} + A\partial_y P,\ \cancel{P\partial_x C} + \cancel{Q\partial_y C}$$
$$+ R\partial_z C + \cancel{A\partial_x R} + \cancel{B\partial_y R} + C\partial_z R + P\partial_z A - \cancel{P\partial_x C} - \cancel{Q\partial_y C} + Q\partial_z B + A\partial_z P - \cancel{A\partial_x R}$$
$$- \cancel{B\partial_y R} + B\partial_z Q)$$

$$= (P\partial_x A + A\partial_x P + Q\partial_x B + R\partial_x C + B\partial_x Q + C\partial_x R,$$
$$Q\partial_y B + B\partial_y Q + R\partial_y C + P\partial_y A + C\partial_y R + A\partial_y P,$$
$$R\partial_z C + C\partial_z R + P\partial_z A + Q\partial_z B + A\partial_z P + B\partial_z Q) = \text{B}$$

$$\therefore\ \nabla(F\cdot G) = (F\cdot\nabla)G + (G\cdot\nabla)F + F\times(\nabla\times G) + G\times(\nabla\times F)$$

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•) $\boxed{\nabla\times(F\times G) = F(\nabla\cdot G) - G(\nabla\cdot F) + (G\cdot\nabla)F - (F\cdot\nabla)G}$

$$\nabla\times(F\times G) = \nabla\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ P & Q & R\\ A & B & C\end{vmatrix} = \nabla\times(QC-BR,\ AR-PC,\ PB-AQ)$$

$$= \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ QC-BR & AR-PC & PB-AQ\end{vmatrix}$$

$$\begin{aligned}= (&P\partial_y B + B\partial_y P - A\partial_y Q - Q\partial_y A - A\partial_z R - R\partial_z A + P\partial_z C + C\partial_z P,\\ &Q\partial_z C + C\partial_z Q - B\partial_z R - R\partial_z B - P\partial_x B - B\partial_x P + A\partial_x Q + Q\partial_x A,\\ &A\partial_x R + R\partial_x A - P\partial_x C - C\partial_x P - Q\partial_y C - C\partial_y Q + B\partial_y R + R\partial_y B) \ldots (0)\end{aligned}$$

Por otro lado:

$$F(\nabla\cdot G) = (P,Q,R)(\partial_x A + \partial_y B + \partial_z C)$$
$$= (P\partial_x A + P\partial_y B + P\partial_z C,\ Q\partial_x A + Q\partial_y B + Q\partial_z C,\ R\partial_x A + R\partial_y B + R\partial_z C)$$

$$G(\nabla\cdot F) = (A,B,C)(\partial_x P + \partial_y Q + \partial_z R)$$
$$= (A\partial_x P + A\partial_y Q + A\partial_z R,\ B\partial_x P + B\partial_y Q + B\partial_z R,\ C\partial_x P + C\partial_y Q + C\partial_z R)$$

y usando resultados del ejercicio anterior.

$$(G\cdot\nabla)F = (A\partial_x P + B\partial_y P + C\partial_z P,\ A\partial_x Q + B\partial_y Q + C\partial_z Q,\ A\partial_x R + B\partial_y R + C\partial_z R)$$

y

$$(F\cdot\nabla)G = (P\partial_x A + Q\partial_y A + R\partial_z A,\ P\partial_x B + Q\partial_y B + R\partial_z B,\ P\partial_x C + Q\partial_y C + R\partial_z C)$$

$\Rightarrow$ $F(\nabla\cdot G) - G(\nabla\cdot F) + (G\cdot\nabla)F - (F\cdot\nabla)G$

$$\begin{aligned}= (&\cancel{P\partial_x A} + P\partial_y B + P\partial_z C - \cancel{A\partial_x P} - A\partial_y Q - A\partial_z R + \cancel{A\partial_x P} + B\partial_y P + C\partial_z P - \cancel{P\partial_x A} - Q\partial_y A\\ &- R\partial_z A,\ Q\partial_x A + \cancel{Q\partial_y B} + Q\partial_z C - B\partial_x P - \cancel{B\partial_y Q} - B\partial_z R + A\partial_x Q + \cancel{B\partial_y Q} + C\partial_z Q\\ &- P\partial_x B - \cancel{Q\partial_y B} - R\partial_z B,\ R\partial_x A + R\partial_y B + \cancel{R\partial_z C} - C\partial_x P - C\partial_y Q - \cancel{C\partial_z R} + A\partial_x R\\ &+ B\partial_y R + \cancel{C\partial_z R} - P\partial_x C - Q\partial_y C - \cancel{R\partial_z C})\end{aligned}$$

$$= (P\partial_y B + P\partial_z C - A\partial_y Q - A\partial_z R + B\partial_y P + C\partial_z P - Q\partial_y A - R\partial_z A,$$
$$Q\partial_x A + Q\partial_z C - B\partial_x P - B\partial_z R + A\partial_x Q + C\partial_z Q - P\partial_x B - R\partial_z B,$$
$$R\partial_x A + R\partial_y B - C\partial_x P - C\partial_y Q + A\partial_x R + B\partial_y R - P\partial_x C - Q\partial_y C) = ⓪$$

$$\therefore\ \nabla \times (F \times G) = F(\nabla \cdot G) - G(\nabla \cdot F) + (G \cdot \nabla)F - (F \cdot \nabla)G$$