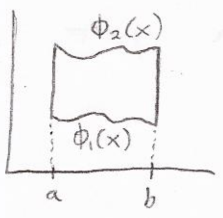
21/08/2014

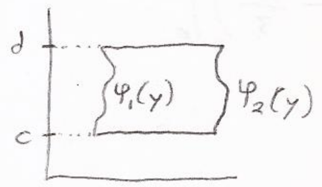
Integrales dobles sobre regiones tipo I y II

Región tipo I



con $\phi_1(x) \leq \phi_2(x) \quad \forall x \in [a,b]$

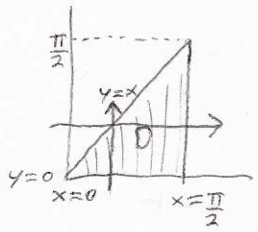
Región tipo II



con $\varphi_1(y) \leq \varphi_2(y) \quad \forall y \in [c,d]$

Ejercicios

1) Sea $f(x,y) = x^3 y + \cos x$ y sea D la región delimitada por $x=0$, $x=\frac{\pi}{2}$, $y=0$ y $y=x$.



Región tipo I $(0 \leq x \leq \frac{\pi}{2},\ 0 \leq y \leq x)$

$$\int_0^{\frac{\pi}{2}}\int_0^x (x^3 y + \cos x)\,dy\,dx = \int_0^{\frac{\pi}{2}} \left(x^3 \left(\frac{y^2}{2}\Big|_0^x \right) + \left(y\Big|_0^x \right)\cos x \right) dx$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{x^5}{2} + x\cos x \right) dx = \left(\frac{x^6}{12} + x\,\mathrm{Sen}\,x + \cos x \right)\Big|_0^{\frac{\pi}{2}}$$

$u = x \quad dv = \cos x\,dx$
$du = dx \quad v = \mathrm{Sen}\,x$

$$= \frac{\pi^6}{2^6(12)} + \frac{\pi}{2} - 1$$

Región tipo II $(0 \leq y \leq \frac{\pi}{2},\ y \leq x \leq \frac{\pi}{2})$

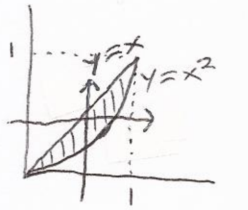
$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_y^{\frac{\pi}{2}} (x^3 y + \cos x)\,dx\,dy = \int_0^{\frac{\pi}{2}} \left(\left(\frac{x^4}{4}\Big|_y^{\frac{\pi}{2}} \right) y + \left(\mathrm{Sen}\,x \Big|_y^{\frac{\pi}{2}} \right) \right) dy$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\pi^4}{2^6} y - \frac{y^5}{4} + 1 - \mathrm{Sen}\,y \right) dy = \left(\frac{\pi^4}{2^6}\frac{y^2}{2} - \frac{y^6}{24} + y + \cos y \right)\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi^6}{8(2^6)} - \frac{\pi^6}{24(2^6)} + \frac{\pi}{2} - 1 = \frac{\pi^6}{2^6(12)} + \frac{\pi}{2} - 1$$

$$\therefore \int_{\substack{\text{Región}\\ \text{tipo I}}} f(x,y)\,dy\,dx = \int_{\substack{\text{Región}\\ \text{tipo II}}} f(x,y)\,dx\,dy$$

2) Sea $f(x,y)=2$ y sea D la región delimitada por $x=0$, $x=1$, $y=x$ y $y=x^2$



Región tipo I. $(0\le x\le 1,\ x^2\le y\le x)$

$$\int_0^1\int_{x^2}^{x} 2\,dy\,dx = 2\int_0^1 (x-x^2)\,dx = 2\left(\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^1$$

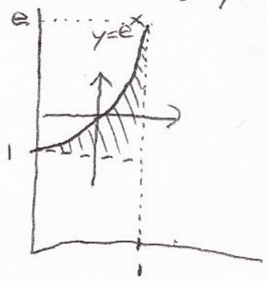
$$= 1-\frac{2}{3} = \frac{1}{3}$$

Región tipo II. $(0\le y\le 1,\ y\le x\le \sqrt{y})$

$$\int_0^1\int_{y}^{\sqrt{y}} 2\,dx\,dy = 2\int_0^1 (\sqrt{y}-y)\,dy = 2\left(\frac{2}{3}y^{3/2}-\frac{y^2}{2}\right)\Big|_0^1 = \frac{4}{3}-1 = \frac{1}{3}$$

$$\therefore \int_{\substack{\text{Región}\\ \text{tipo I}}} f(x,y)\,dy\,dx = \int_{\substack{\text{Región}\\ \text{tipo II}}} f(x,y)\,dx\,dy$$

3) Sea $f(x,y)=x+y$ y sea D la región delimitada por $x=0$, $x=1$, $y=1$ y $y=e^x$



Región tipo I. $(0\le x\le 1,\ 1\le y\le e^x)$

$$\int_0^1\int_1^{e^x} (x+y)\,dy\,dx = \int_0^1 \left(x\left(y\Big|_1^{e^x}\right) + \left(\frac{y^2}{2}\Big|_1^{e^x}\right)\right)dx$$

$$= \int_0^1 \left(xe^x - x + \frac{e^{2x}}{2} - \frac{1}{2}\right)dx$$

$u=x \quad dv=e^x dx$
$du=dx \quad v=e^x$

$$= \left(xe^x - e^x - \frac{x^2}{2} + \frac{e^{2x}}{4} - \frac{x}{2}\right)\Big|_0^1 = \cancel{e}-\cancel{e}-\cancel{\tfrac{1}{2}}+\frac{e^2}{4}-\cancel{\tfrac{1}{2}}+\cancel{1}-\frac{1}{4}$$

$$= \frac{1}{4}(e^2-1)$$

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Región tipo II. ($1 \leq y \leq e$, $\ln y \leq x \leq 1$)

$$\int_1^e \int_{\ln y}^1 (x+y)\,dx\,dy = \int_1^e \left(\left(\frac{x^2}{2}\Big|_{\ln y}^1\right) + \left(x\Big|_{\ln y}^1\right)y\right)dy$$

$$= \int_1^e \left(\frac{1}{2} - \frac{(\ln y)^2}{2} + y - y\ln y\right)dy$$

$u = \ln y \quad dv = \ln y\,dy$
$du = \frac{1}{y}dy \quad v = y\ln y - y$

$r = \ln y \quad dw = y\,dy$
$dr = \frac{1}{y}dy \quad w = \frac{y^2}{2}$

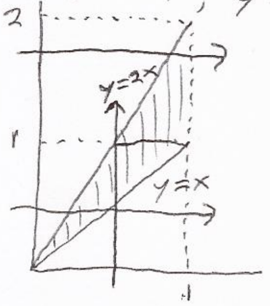
$$= \left(\frac{y}{2} - \frac{1}{2}\left(y(\ln y)^2 - y\ln y - y\ln y + y + y\right) + \frac{y^2}{2} - \left(\frac{y^2}{2}\ln y - \frac{1}{2}\frac{y^2}{2}\right)\right)\Bigg|_1^e$$

$$= \frac{e}{2} - \frac{1}{2}(e - \cancel{e} - \cancel{e} + 2e) + \frac{e^2}{2} - \left(\frac{e^2}{2} - \frac{e^2}{4}\right) - \cancel{\frac{1}{2}} + \cancel{1} - \cancel{\frac{1}{2}} - \frac{1}{4}$$

$$= \cancel{\frac{e}{2}} - \cancel{\frac{e}{2}} + \cancel{\frac{e^2}{2}} - \cancel{\frac{e^2}{2}} + \frac{e^2}{4} - \frac{1}{4} = \frac{1}{4}(e^2 - 1)$$

$$\therefore \int_{\substack{\text{Región}\\ \text{tipo I}}} f(x,y)\,dy\,dx = \int_{\substack{\text{Región}\\ \text{tipo II}}} f(x,y)\,dx\,dy$$

4) Sea $f(x,y) = \operatorname{sen} y$ y sea D la región delimitada por $x=0$, $x=1$, $y=x$, $y=2x$.



Región tipo I. ($0 \leq x \leq 1$, $x \leq y \leq 2x$)

$$\int_0^1 \int_x^{2x} \operatorname{sen} y\,dy\,dx = -\int_0^1 \left(\cos y\Big|_x^{2x}\right)dx$$

$$= -\int_0^1 (\cos 2x - \cos x)\,dx = -\left(\frac{1}{2}\operatorname{sen} 2x - \operatorname{sen} x\right)\Big|_0^1$$

$$= \operatorname{sen}(1) - \frac{1}{2}\operatorname{sen}(2)$$

Región tipo II. (Hay que considerarla en 2 partes).

① ($0 \leq y \leq 1$, $\frac{y}{2} \leq x \leq y$)

$$\int_0^1 \int_{\frac{y}{2}}^{y} \operatorname{sen} y\,dx\,dy = \int_0^1 \left(y \operatorname{sen} y - \frac{y}{2}\operatorname{sen} y\right)dy = \int_0^1 \frac{y}{2}\operatorname{sen} y\,dy$$

$u = y \quad dv = \operatorname{sen} y\,dy$
$du = dy \quad v = -\cos y$

$= \frac{1}{2}(-y\cos y + \operatorname{sen} y)\Big|_0^1 = -\frac{1}{2}\cos(1) + \frac{1}{2}\operatorname{sen}(1)$

Ahora ② $(1 \leq y \leq 2,\ \frac{y}{2} \leq x \leq 1)$

$$\int_1^2 \int_{\frac{y}{2}}^1 \operatorname{sen} y \, dx\, dy = \int_1^2 (\operatorname{sen} y - \frac{y}{2}\operatorname{sen} y)\, dy$$

$u = y \quad dv = \operatorname{sen} y\, dy$
$du = dy \quad v = -\cos y$

$$= \left(-\cos y - \frac{1}{2}(-y\cos y + \operatorname{sen} y)\right)\Big|_1^2 = -\cancel{\cos(2)} + \cancel{\cos(2)} - \frac{1}{2}\operatorname{sen}(2) + \cos(1) - \frac{1}{2}\cos(1) + \frac{1}{2}\operatorname{sen}(1)$$

$$= \frac{1}{2}\cos(1) + \frac{1}{2}\operatorname{sen}(1) - \frac{1}{2}\operatorname{sen}(2)$$

∴ Si ahora sumamos ①+② $= -\frac{1}{2}\cancel{\cos}(1) + \frac{1}{2}\operatorname{sen}(1) + \frac{1}{2}\cancel{\cos}(1) + \frac{1}{2}\operatorname{sen}(1) - \frac{1}{2}\operatorname{sen}(2)$

$$= \operatorname{sen}(1) - \frac{1}{2}\operatorname{sen}(2)$$

$$\therefore \int_{\text{Región tipo I}} f(x,y)\,dy\,dx = \int_{\text{Región tipo II}} f(x,y)\,dx\,dy$$