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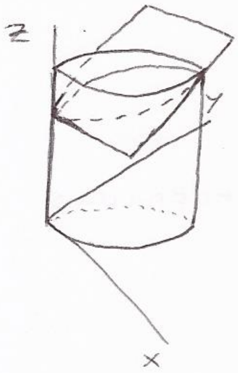
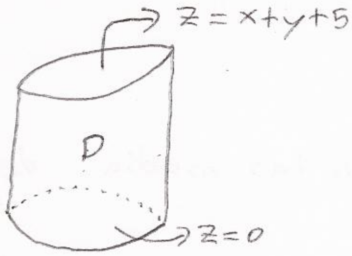
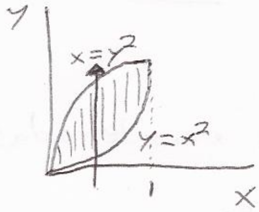
Integrales de volumen, CM y centroides

1) Calcular el volumen del sólido en $\mathbb{R}^3$ delimitado por $y=x^2$, $x=y^2$, $z=x+y+5$ y $z=0$



i,e





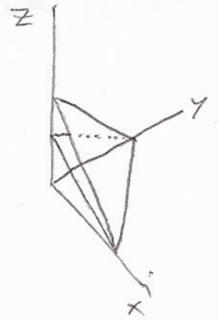
$$\Rightarrow \text{Vol}(D)=\iiint_D 1\,dv=\int_0^1\int_{x^2}^{\sqrt{x}}\int_0^{x+y+5} 1\,dz\,dy\,dx=\int_0^1\int_{x^2}^{\sqrt{x}}(x+y+5)\,dy\,dx$$

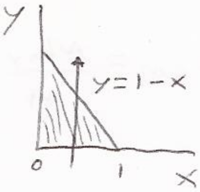
$$=\int_0^1\left(x\left(y\Big|_{x^2}^{\sqrt{x}}\right)+\left(\frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}}\right)+5\left(y\Big|_{x^2}^{\sqrt{x}}\right)\right)dx$$

$$=\int_0^1\left(x^{3/2}-x^3+\frac{x}{2}-\frac{x^4}{2}+5x^{1/2}-5x^2\right)dx=\left(\frac{2}{5}x^{5/2}-\frac{x^4}{4}+\frac{x^2}{4}-\frac{x^5}{10}+\frac{10}{3}x^{3/2}-\frac{5}{3}x^3\right)\Bigg|_0^1$$

$$=\frac{2}{5}-\cancel{\frac{1}{4}}+\cancel{\frac{1}{4}}-\frac{1}{10}+\frac{10}{3}-\frac{5}{3}=\frac{2}{5}-\frac{1}{10}+\frac{5}{3}=\underline{\frac{59}{30}}$$

2) Calcular el volumen del sólido en $\mathbb{R}^3$ delimitado por $x+y+z=1$ y $x+y+2z=1$ en el octante $x\geq 0$, $y\geq 0$ y $z\geq 0$





$$\Rightarrow \frac{1-x-y}{2}\leq z\leq 1-x-y$$
$$0\leq y\leq 1-x$$
$$0\leq x\leq 1$$

$$\Rightarrow \text{Vol}(D)=\int_0^1\int_0^{1-x}\int_{\frac{1-x-y}{2}}^{1-x-y} 1\,dz\,dy\,dx=\int_0^1\int_0^{1-x}\left(1-x-y-\frac{1}{2}+\frac{x}{2}+\frac{y}{2}\right)dy\,dx$$

$$=\int_0^1\int_0^{1-x}\left(\frac{1}{2}-\frac{x}{2}-\frac{y}{2}\right)dy\,dx=\int_0^1\left(\frac{1}{2}\left(y\Big|_0^{1-x}\right)-\frac{x}{2}\left(y\Big|_0^{1-x}\right)-\frac{1}{2}\left(\frac{y^2}{2}\Big|_0^{1-x}\right)\right)dx$$

$$=\int_0^1\left(\frac{1-x}{2}-\frac{x}{2}(1-x)-\frac{1}{4}(1-x)^2\right)dx=\int_0^1\left(\frac{1}{2}-\frac{x}{2}-\frac{x}{2}+\frac{x^2}{2}-\frac{1}{4}(1-x)^2\right)dx$$

$$= \int_0^1 \left(\frac{1}{2} - x + \frac{x^2}{2} - \frac{1}{4}(1-x)^2\right)dx = \left(\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{4}\frac{(1-x)^3}{3}\right)\Big|_0^1$$

$u = 1-x$
$du = -dx$

$$= \cancel{\frac{1}{2}} - \cancel{\frac{1}{2}} + \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$$

Centroide

Es el pto. cuyas coordenadas son las medias de las coordenadas sobre el dominio.

En $\mathbb{R}^2$ $\quad \bar{x} = \frac{1}{A}\iint_D x\,dA$, $\bar{y} = \frac{1}{A}\iint_D y\,dA$ donde $A = \iint_D 1\,dA$

En $\mathbb{R}^3$ $\quad \bar{x} = \frac{1}{V}\iiint_D x\,dV$, $\bar{y} = \frac{1}{V}\iiint_D y\,dV$, $\bar{z} = \frac{1}{V}\iiint_D z\,dV$ donde $V = \iiint_D 1\,dV$

CM

Es el pto. (que puede estar dentro o fuera del cuerpo) que determina el centro de la masa de un objeto en base a una distribución o densidad de masa ρ.

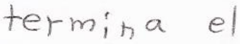
En $\mathbb{R}^2$ $\quad M_y = \iint_D x\rho\,dA$, $M_x = \iint_D y\rho\,dA$ y $M = \iint_D \rho\,dA$

$$x_{CM} = \frac{M_y}{M}, \quad y_{CM} = \frac{M_x}{M}$$

En $\mathbb{R}^3$ $\quad M_{yz} = \iiint_D x\rho\,dV$, $M_{xz} = \iiint_D y\rho\,dV$, $M_{xy} = \iiint_D z\rho\,dV$ y $M = \iiint_D \rho\,dV$

$$x_{CM} = \frac{M_{yz}}{M}, \quad y_{CM} = \frac{M_{xz}}{M}, \quad z_{CM} = \frac{M_{xy}}{M}$$

Ejemplos

①  varilla de longitud L con $\rho(x) = 1$

Centroide

$$L = \int_0^L 1\,dx \Rightarrow \bar{x} = \frac{1}{L}\int_0^L x\,dx = \frac{1}{L}\left(\frac{x^2}{2}\Big|_0^L\right) = \frac{1}{L}\left(\frac{L^2}{2}\right) = \frac{L}{2}$$

CM

$$M = \int_0^L \rho(x)\,dx = \int_0^L 1\,dx = L \quad \text{y} \quad M_x = \int_0^L x\rho(x)\,dx = \int_0^L x\,dx = \frac{L^2}{2}$$

$$\Rightarrow x_{CM} = \frac{\frac{L^2}{2}}{L} = \frac{L}{2} \qquad \therefore \ \bar{x} = x_{CM}$$

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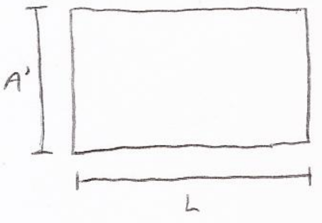
si $\rho(x)=\frac{1}{x}$ $\Rightarrow$ $\bar{x}$ no cambia, i,e $\bar{x}=\frac{L}{2}$, pero

$M_x=\int_0^L x\frac{1}{x}dx = L$ y $M=\int_0^L \frac{1}{x}dx = \ln L - \ln 0 \to \infty$

$\therefore x_{cm}\to 0$ $\therefore$ $\underline{\bar{x}\neq x_{cm}}$

② 

con $\rho(x,y)=1$

centroide

$A=\int_0^L\int_0^{A'}1\,dy\,dx = LA'$ $\Rightarrow$ $\bar{x}=\frac{1}{LA'}\int_0^L\int_0^{A'}x\,dy\,dx=\frac{1}{LA'}\int_0^L xA'dx=\frac{1}{LA'}\left(\frac{L^2A'}{2}\right)=\frac{L}{2}$

$\bar{y}=\frac{1}{LA'}\int_0^L\int_0^{A'}y\,dy\,dx=\frac{1}{LA'}\int_0^L\frac{A'^2}{2}dx=\frac{1}{LA'}\left(\frac{LA'^2}{2}\right)=\frac{A'}{2}$

$\therefore$ $\underline{(\bar{x},\bar{y})=\frac{1}{2}(L,A')}$

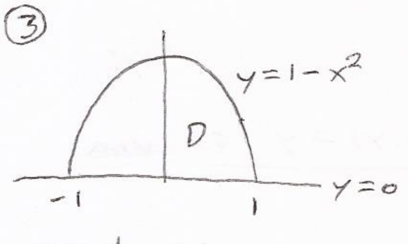
CM

$M=\int_0^L\int_0^{A'}\rho(x,y)\,dy\,dx = LA'$, $M_y=\int_0^L\int_0^{A'}x\rho\,dy\,dx=\int_0^L xA'dx=\frac{L^2A'}{2}$

$M_x=\int_0^L\int_0^{A'}y\rho\,dy\,dx=\int_0^L\frac{A'^2}{2}dx=\frac{LA'^2}{2}$

$\therefore x_{cm}=\frac{M_y}{M}=\frac{L}{2}$, $y_{cm}=\frac{M_x}{M}=\frac{A'}{2}$

$\therefore$ $\underline{(x_{cm},y_{cm})=\frac{1}{2}(L,A')=(\bar{x},\bar{y})}$

③ 

$\rho(x,y)=y$

centroide

$A=\int_{-1}^1\int_0^{1-x^2}1\,dy\,dx = \int_{-1}^1(1-x^2)dx=\left(x-\frac{x^3}{3}\right)\Big|_{-1}^1 = 1-\frac{1}{3}-\left(-1+\frac{1}{3}\right)=2-\frac{2}{3}=\frac{4}{3}$

$\bar{x}=\frac{3}{4}\int_{-1}^1\int_0^{1-x^2}x\,dy\,dx=\frac{3}{4}\int_{-1}^1 x\left(y\Big|_0^{1-x^2}\right)dx=\frac{3}{4}\int_{-1}^1(x-x^3)dx=\frac{3}{4}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)\Big|_{-1}^1=\frac{1}{2}-\frac{1}{4}-\frac{1}{2}+\frac{1}{4}$

$\therefore \bar{x}=0$

$$\bar{y} = \frac{3}{4}\int_{-1}^{1}\int_{0}^{1-x^2} y\,dy\,dx = \frac{3}{4}\int_{-1}^{1}\frac{(1-x^2)^2}{2}dx = \frac{3}{8}\int_{-1}^{1}(1-2x^2+x^4)\,dx = \frac{3}{8}\left(x-\frac{2}{3}x^3+\frac{x^5}{5}\right)\Big|_{-1}^{1}$$

$$= \frac{3}{8}\left(1-\frac{2}{3}+\frac{1}{5}-\left(-1+\frac{2}{3}-\frac{1}{5}\right)\right) = \frac{3}{8}\left(2-\frac{4}{3}+\frac{2}{5}\right) = \frac{2}{5}$$

$$\therefore\ (\bar{x},\bar{y}) = \left(0,\frac{2}{5}\right)$$

CM

$$M = \int_{-1}^{1}\int_{0}^{1-x^2} y\,dy\,dx = \frac{1}{2}\left(2-\frac{4}{3}+\frac{2}{5}\right) = \frac{1}{2}\left(\frac{16}{15}\right) = \frac{8}{15}$$

pues ya se calculó arriba la misma integral salvo por el 3/4

$$M_x = \int_{-1}^{1}\int_{0}^{1-x^2} y^2\,dy\,dx = \int_{-1}^{1}\frac{(1-x^2)^3}{3}dx = \frac{1}{3}\int_{-1}^{1}(1-3x^2+3x^4-x^6)\,dx$$

$$= \frac{1}{3}\left(x-x^3+\frac{3}{5}x^5-\frac{x^7}{7}\right)\Big|_{-1}^{1} = \frac{1}{3}\left(\cancel{1}-\cancel{1}+\frac{3}{5}-\frac{1}{7}-\left(-\cancel{1}+\cancel{1}-\frac{3}{5}+\frac{1}{7}\right)\right)$$

$$= \frac{1}{3}\left(\frac{6}{5}-\frac{2}{7}\right) = \frac{32}{105}$$

$$\therefore\ y_{cm} = \frac{M_x}{M} = \frac{32}{105}\left(\frac{15}{8}\right) = \frac{4}{7}$$

Para calcular x_{cm} nos fijamos en que D es simétrico respecto al eje y, así como también ρ, i,e $\rho(-x,y)=\rho(x,y)=y$

$$\therefore\ x_{cm} = 0$$

$$\therefore\ (x_{cm}, y_{cm}) = \left(0,\frac{4}{7}\right)$$

$\therefore\ (x_{cm}, y_{cm}) \neq (\bar{x},\bar{y})$ pues $\rho(x,y)=y$ es una función variable

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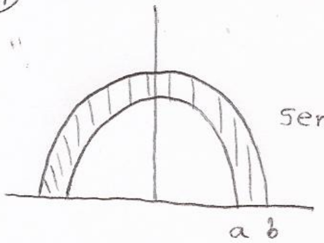

semianillo con $\rho(x,y)=1$

Trabajaremos con coordenadas polares $x=r\cos\theta$, $y=r\,\text{sen}\,\theta$

centroide

$$A=\int_0^{\pi}\int_a^{b} r\,dr\,d\theta=\int_0^{\pi}\frac{b^2-a^2}{2}\,d\theta=\frac{\pi}{2}(b^2-a^2)$$

(r: Jacobiano debido a cambio a polares)

$$\bar{x}=\frac{2}{\pi(b^2-a^2)}\int_0^{\pi}\int_a^{b} r^2\cos\theta\,dr\,d\theta=\frac{2}{\pi(b^2-a^2)}\int_0^{\pi}\frac{b^3-a^3}{3}\cos\theta\,d\theta=\frac{2(b^3-a^3)}{3\pi(b^2-a^2)}\,\text{sen}\,\theta\Big|_0^{\pi}=0$$

$$\bar{y}=\frac{2}{\pi(b^2-a^2)}\int_0^{\pi}\int_a^{b} r^2\,\text{sen}\,\theta\,dr\,d\theta=\frac{2}{\pi(b^2-a^2)}\int_0^{\pi}\frac{b^3-a^3}{3}\,\text{sen}\,\theta\,d\theta=\frac{2(a^3-b^3)}{3\pi(b^2-a^2)}\cos\theta\Big|_0^{\pi}$$

$$=\frac{4}{3\pi}\frac{(b^3-a^3)}{(b^2-a^2)}$$

$$\therefore\ (\bar{x},\bar{y})=\left(0,\frac{4}{3\pi}\frac{(b^3-a^3)}{(b^2-a^2)}\right)$$

CM

$$M=\int_0^{\pi}\int_a^{b} r\,dr\,d\theta=\frac{\pi}{2}(b^2-a^2)$$ (igual que A, ya que $\rho(x,y)=1$)

$$M_x=\int_0^{\pi}\int_a^{b} r^2\,\text{sen}\,\theta\,dr\,d\theta=\int_0^{\pi}\frac{b^3-a^3}{3}\,\text{sen}\,\theta\,d\theta=-\left(\frac{b^3-a^3}{3}\right)\cos\theta\Big|_0^{\pi}=\frac{2}{3}(b^3-a^3)$$

$$\therefore\ y_{cm}=\frac{M_x}{M}=\frac{\frac{2}{3}(b^3-a^3)}{\frac{\pi}{2}(b^2-a^2)}=\frac{4}{3\pi}\frac{(b^3-a^3)}{(b^2-a^2)}$$

y nuevamente, el semianillo es simétrico respecto al eje y y $\rho(-x,y)=\rho(x,y)=1$ $\therefore\ x_{cm}=0$

$$\therefore\ (x_{cm},y_{cm})=\left(0,\frac{4}{3\pi}\frac{(b^3-a^3)}{(b^2-a^2)}\right)=(\bar{x},\bar{y})$$