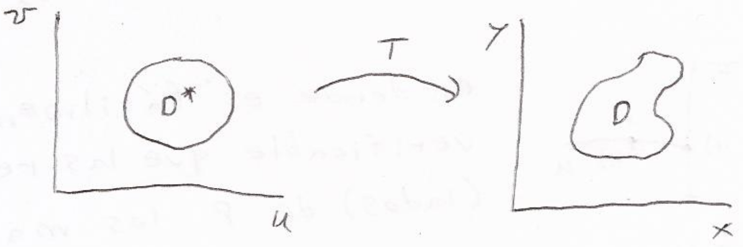
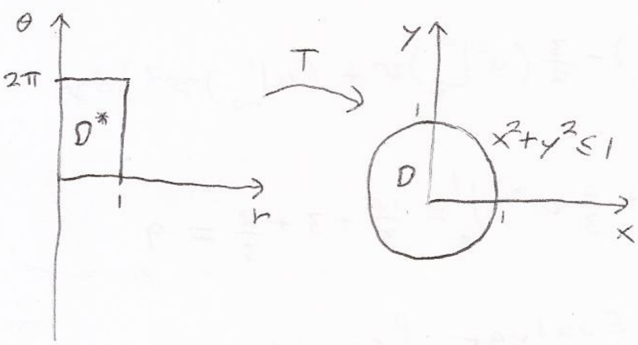
28/08/2014

Cambio de variable



$$\int_D f(x,y)\,dxdy = \int_{D^*} f(x(u,v),\, y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dudv$$

① Sea $T: D^* \to D$ donde $D = T(D^*)$ es el conjunto de $(x,y) \in \mathbb{R}^2$ $\ni$ $x^2+y^2 \leq 1$ y $D^* = [0,1]\times[0,2\pi]$ y $T(r,\theta) = (r\cos\theta, r\operatorname{sen}\theta)$
Calcular el área(D).

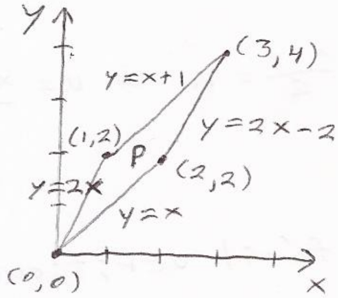


$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\operatorname{sen}\theta \\ \operatorname{sen}\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\operatorname{sen}^2\theta = r$$

$$\Rightarrow \int_D dxdy = \int_0^{2\pi}\int_0^1 \left| \frac{\partial(x,y)}{\partial(r,\theta)} \right| drd\theta = \int_0^{2\pi}\int_0^1 r\,drd\theta = \frac{1}{2}\int_0^{2\pi} d\theta = \frac{2\pi}{2} = \pi$$

② Sea P el paralelogramo acotado por $y = 2x$, $y = 2x-2$, $y = x$ y $y = x+1$. Evaluar $\int_P xy\,dxdy$ haciendo el cambio de variables $x = u - v$, $y = 2u - v$, i,e $T(u,v) = (u-v, 2u-v)$.



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} = 1$$

Poniendo a u, v como funciones de x, y

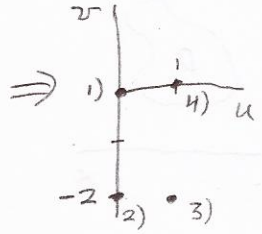
$\Rightarrow$ $x + v = \frac{y+v}{2}$ $\Rightarrow$ $\frac{v}{2} = \frac{y}{2} - x$ $\Rightarrow$ $v = y - 2x$

y $x + v = x + y - 2x = y - x = u$ $\Rightarrow$ $u = y - x$

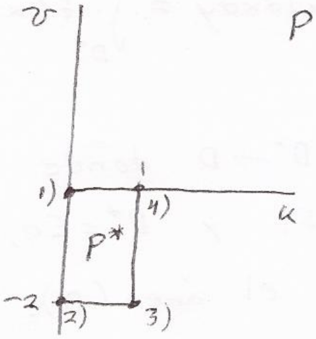
$\Rightarrow$ evaluando los vértices de P para ponerlos en coordenadas u, v tenemos que

1) $(0,0) \Rightarrow u=0,\ v=0$
2) $(2,2) \Rightarrow u=0,\ v=-2$
3) $(3,4) \Rightarrow u=1,\ v=-2$
4) $(1,2) \Rightarrow u=1,\ v=0$

$\Rightarrow$



en donde es fácilmente verificable que las rectas (lados) de P los manda al rectángulo P^*

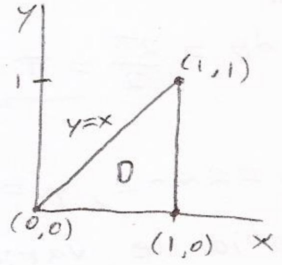


$$\Rightarrow \int_P xy\,dx\,dy = \int_{P^*} (u-v)(2u-v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$\Rightarrow \int_0^1\int_{-2}^0 (2u^2-3uv+v^2)\,du\,dv = \int_0^1 \left(\frac{2}{3}\left(u^3\Big|_{-2}^0\right) - \frac{3}{2}\left(u^2\Big|_{-2}^0\right)v + \left(u\Big|_{-2}^0\right)v^2\right) dv$$

$$= \int_0^1 \left(\frac{16}{3} + 6v + 2v^2\right) dv = \left(\frac{16}{3}v + 3v^2 + \frac{2}{3}v^3\right)\Big|_0^1 = \frac{16}{3} + 3 + \frac{2}{3} = 9$$

③ Sea D la región $0 \leq y \leq x$, $0 \leq x \leq 1$. Evaluar $\int_D (x+y)\,dx\,dy$ haciendo $x = u+v$ y $y = u-v$.



$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

Poniendo u, v en términos de x, y

$x - v = y + v \Rightarrow v = \frac{x-y}{2}$

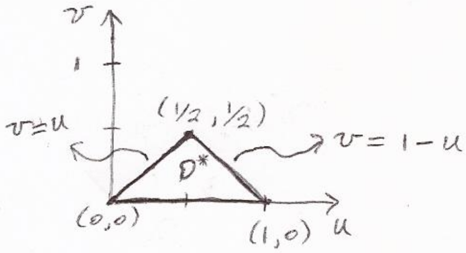
$\Rightarrow x - v = x - \frac{x}{2} + \frac{y}{2} = \frac{x}{2} + \frac{y}{2} = \frac{x+y}{2} = u \Rightarrow u = \frac{x+y}{2}$

$\Rightarrow$ los vértices de D los manda a:

1) $(0,0) \Rightarrow u=0,\ v=0$
2) $(1,0) \Rightarrow u=v=\frac{1}{2}$
3) $(1,1) \Rightarrow u=1,\ v=0$

en donde también es fácil verificar que los lados de D los manda a los lados de otro triángulo D^* con vértices $(0,0)$, $\left(\frac{1}{2},\frac{1}{2}\right)$ y $(1,0)$

28/08/2014

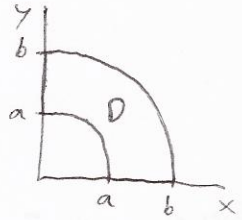


$$\Rightarrow \int_0^{\frac12}\int_v^{1-v} 2(u+v+u-v)\,du\,dv = \int_0^{1/2}\int_v^{1-v} 2(2u)\,du\,dv$$

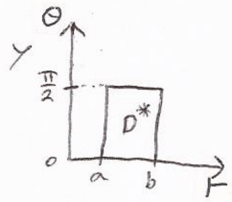
$$= 4\int_0^{1/2}\left(\frac{u^2}{2}\Big|_v^{1-v}\right)dv = 2\int_0^{\frac12}(1-2v+\cancel{v^2}-\cancel{v^2})\,dv$$

$$= 2\int_0^{\frac12}(1-2v)\,dv = 2(v-v^2)\Big|_0^{\frac12} = 2\left(\frac12-\frac14\right) = \frac12$$

④ Evaluar $\int_D \ln(x^2+y^2)\,dx\,dy$, donde D es la región en el 1er. cuadrante que está entre los arcos de los círculos $x^2+y^2=b^2$ y $x^2+y^2=a^2$ con $0<a<b$



Haciendo cambio a polares con $a\le r\le b$, $0\le\theta\le\frac{\pi}{2}$ y recordando que $\frac{\partial(x,y)}{\partial(r,\theta)}=r$



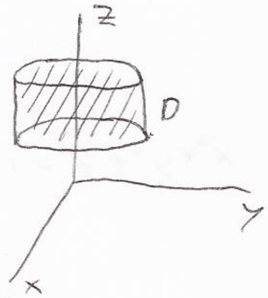
$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_a^b r\ln(r^2)\,dr\,d\theta$$ ya que $x^2+y^2=r^2$

$u=r^2$
$du=2r\,dr$

$$\Rightarrow \int_0^{\frac{\pi}{2}}\int_a^b r\ln(r^2)\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\int_a^b \frac12 \ln u\,du\,d\theta = \frac12\int_0^{\frac{\pi}{2}}(r^2\ln r^2 - r^2)\Big|_a^b\,d\theta$$

$$= \frac{\pi}{4}(b^2\ln b^2 - b^2 - a^2\ln a^2 + a^2) = \frac{\pi}{2}\left(b^2\ln b - a^2\ln a - \frac12(b^2-a^2)\right)$$

⑤ Integrar $ze^{x^2+y^2}$ sobre $x^2+y^2\le 4$, $2\le z\le 3$



Usando coordenadas cilíndricas

$x = r\cos\theta$
$y = r\,\mathrm{sen}\,\theta$
$z = z$

$$\Rightarrow \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\,\mathrm{sen}\,\theta & 0 \\ \mathrm{sen}\,\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\,\mathrm{sen}^2\theta = r$$

$$\Rightarrow \int_2^3\int_0^{2\pi}\int_0^2 z\,r\,e^{r^2}\,dr\,d\theta\,dz = 2\pi\left(\frac{z^2}{2}\Big|_2^3\right)\int_0^2 r e^{r^2}\,dr = 5\pi\int_0^2 r e^{r^2}\,dr$$

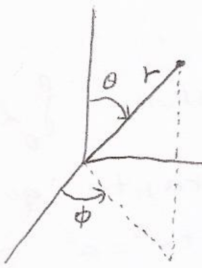
$u=r^2$
$du=2r\,dr$

$$= \frac{5\pi}{2}\int_0^2 e^u du = \frac{5\pi}{2} e^{r^2}\Big|_0^2 = \frac{5\pi}{2}(e^4-1)$$

⑥ Evaluar $\int_D e^{(x^2+y^2+z^2)^{3/2}} dV$ donde D es la bola unitaria en $\mathbb{R}^3$

Usando coordenadas esféricas

$x = r\,\text{sen}\,\theta \cos\phi$
$y = r\,\text{sen}\,\theta\,\text{sen}\,\phi$
$z = r\cos\theta$



$$\Rightarrow \frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} = \begin{vmatrix} \text{sen}\,\theta\cos\phi & -r\,\text{sen}\,\theta\,\text{sen}\,\phi & r\cos\theta\cos\phi \\ \text{sen}\,\theta\,\text{sen}\,\phi & r\,\text{sen}\,\theta\cos\phi & r\cos\theta\,\text{sen}\,\phi \\ \cos\theta & 0 & -r\,\text{sen}\,\theta \end{vmatrix}$$

$$= \text{sen}\,\theta\cos\phi(-r^2\text{sen}^2\theta\cos\phi) + r\,\text{sen}\,\theta\,\text{sen}\,\phi(-r\,\text{sen}^2\theta\,\text{sen}\,\phi - r\cos^2\theta\,\text{sen}\,\phi)$$
$$+ r\cos\theta\cos\phi(-r\,\text{sen}\,\theta\cos\theta\cos\phi)$$

$$= -r^2\text{sen}^3\theta\cos^2\phi - r^2\text{sen}^3\theta\,\text{sen}^2\phi - r^2\text{sen}\,\theta\cos^2\theta\cos^2\phi - r^2\text{sen}\,\theta\cos^2\theta\,\text{sen}^2\phi$$

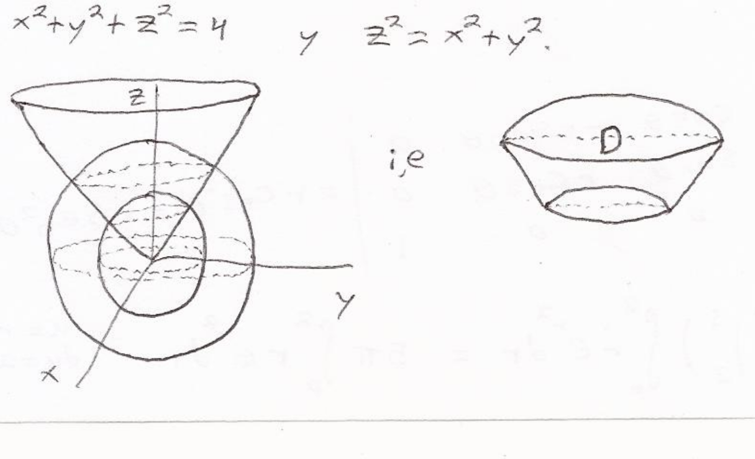
$$= -r^2\text{sen}^3\theta - r^2\text{sen}\,\theta\cos^2\theta = -r^2\text{sen}\,\theta$$

$$\therefore \int_D e^{(x^2+y^2+z^2)^{3/2}} dV = \int_0^\pi\int_0^{2\pi}\int_0^1 r^2\text{sen}\,\theta\, e^{r^3} dr\,d\phi\,d\theta = 2\pi\int_0^\pi\int_0^1 r^2\text{sen}\,\theta\, e^{r^3} dr\,d\theta$$

$$= -2\pi\left(\cos\theta\Big|_0^\pi\right)\int_0^1 r^2 e^{r^3} dr = \frac{4\pi}{3}\int_0^1 e^u du = \frac{4\pi}{3}(e-1)$$

$u = r^3$
$du = 3r^2 dr$

⑦ Integrar $\frac{1}{x^2+y^2+z^2}$ sobre el sólido acotado por $x^2+y^2+z^2=1$, $x^2+y^2+z^2=4$ y $z^2 = x^2+y^2$.



donde $0 \le \theta \le \frac{\pi}{4}$

$0 \le \phi \le 2\pi$

$1 \le r \le 2$

28/08/2014

$$\Rightarrow \int_D \frac{dV}{x^2+y^2+z^2} = \int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_1^2 \frac{1}{\cancel{r^2}}\cancel{r^2}\,\text{sen}\,\theta\, dr\, d\phi\, d\theta = \int_0^{\frac{\pi}{4}}\int_0^{2\pi} \text{sen}\,\theta\, d\phi\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \text{sen}\,\theta\, d\theta = -2\pi\left(\cos\theta \Big|_0^{\frac{\pi}{4}}\right) = -2\pi\left(\frac{\sqrt{2}}{2} - 1\right) = \pi\left(2 - \sqrt{2}\right)$$