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Derivación bajo integral

Calcular la integral:

$$\int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + a^2 \cos^2 x)\,dx \qquad a \neq 0$$

Sol.

Para esto definimos $f(x,y) = \ln(\operatorname{sen}^2 x + y^2 \cos^2 x)$ la cual es derivable y continua. Definimos

$$g(y) = \int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + y^2 \cos^2 x)\,dx$$

$$\Rightarrow g'(y) = \frac{\partial}{\partial y}\int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + y^2 \cos^2 x)\,dx = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y}\left(\ln(\operatorname{sen}^2 x + y^2 \cos^2 x)\right)dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2y\cos^2 x}{\operatorname{sen}^2 x + y^2\cos^2 x}\,dx = \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{(y^2-1)\cos^2 x}{\operatorname{sen}^2 x + y^2\cos^2 x}\,dx$$

$$= \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{y^2\cos^2 x - \cos^2 x}{\operatorname{sen}^2 x + y^2\cos^2 x}\,dx = \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{\operatorname{sen}^2 x - \operatorname{sen}^2 x + y^2\cos^2 x - \cos^2 x}{\operatorname{sen}^2 x + y^2\cos^2 x}\,dx$$

$$= \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}}\left(1 - \frac{1}{\operatorname{sen}^2 x + y^2\cos^2 x}\right)dx = \frac{2y}{y^2-1}\left(\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{dx}{\operatorname{sen}^2 x + y^2\cos^2 x}\right) \cdots ①$$

donde

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\operatorname{sen}^2 x + y^2\cos^2 x} = \int_0^{\frac{\pi}{2}} \frac{dx}{\cos^2 x(\tan^2 x + y^2)} = \int_0^{\infty} \frac{du}{u^2+y^2}$$

$u = \tan x$ $\qquad$ $u = y\tan\theta$

$du = \sec^2 x\,dx$ $\qquad$ $du = y\sec^2\theta\,d\theta$

$$= \int_0^{\infty} \frac{y\sec^2\theta\,d\theta}{y^2(\tan^2\theta+1)} = \frac{1}{y}\int_0^{\infty} d\theta = \frac{1}{y}\theta = \frac{1}{y}\arctan\left(\frac{u}{y}\right)\Big|_0^{\infty} = \frac{1}{y}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2y}$$

$\therefore$ de ①

$$g'(y) = \frac{2y}{y^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2y}\right) = \frac{\pi y}{y^2-1}\left(1 - \frac{1}{y}\right) = \frac{\pi}{y^2-1}(y-1) = \frac{\pi}{y+1}$$

$\therefore\ g'(y) = \dfrac{\pi}{y+1} \Rightarrow g(y) = \pi \ln(y+1) + C$

Ahora, como $g(y) = \int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + y^2 \cos^2 x)\,dx \Rightarrow g(1) = 0$

$\Rightarrow g(1) = \pi \ln 2 + C = 0 \Rightarrow C = -\pi \ln 2$

de esta manera:

$$g(y) = \pi \ln(y+1) - \pi \ln 2 \Rightarrow g(y) = \pi \ln\left(\frac{y+1}{2}\right)$$

y regresando a la integral

$$\int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + y^2 \cos^2 x)\,dx = \pi \ln\left(\frac{y+1}{2}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \ln(\operatorname{sen}^2 x + a^2 \cos^2 x)\,dx = \pi \ln\left(\frac{a+1}{2}\right)$$