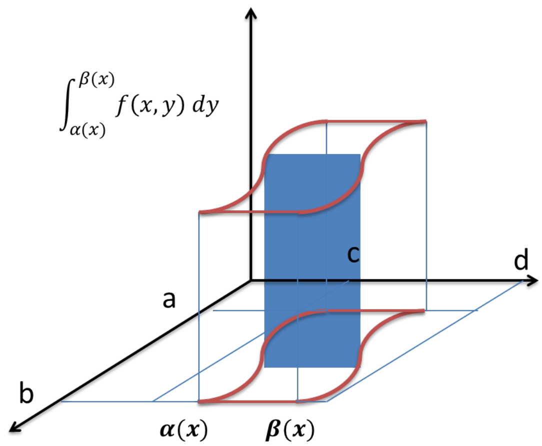
Teorema 1. *Teorema de Leibniz Sea $f:[a,b]\times[c,d]\subset\mathbb{R}^2\to\mathbb{R}$ una función continua y sean $\alpha,\beta:[a,b]\to\mathbb{R}$ funciones derivables tales que*

$$a\leq x\leq b$$

$$c\leq\alpha(x)\leq y\leq\beta(x)\leq d \quad \forall \quad y\in[c,d]$$



supongamos que $\dfrac{\partial f}{\partial x}$ *existe y es continua en el conjunto*

$$T=\{(x,y)\mathbb{R}^2 \mid \alpha(x)\leq y\leq\beta(x), \quad x\in[a,b]\}$$

entonces

$$F(x)=\int_{\alpha(x)}^{\beta(x)} f(x,y)\ dy$$

existe es derivable $\forall\ y\in[c,d]$ *y*

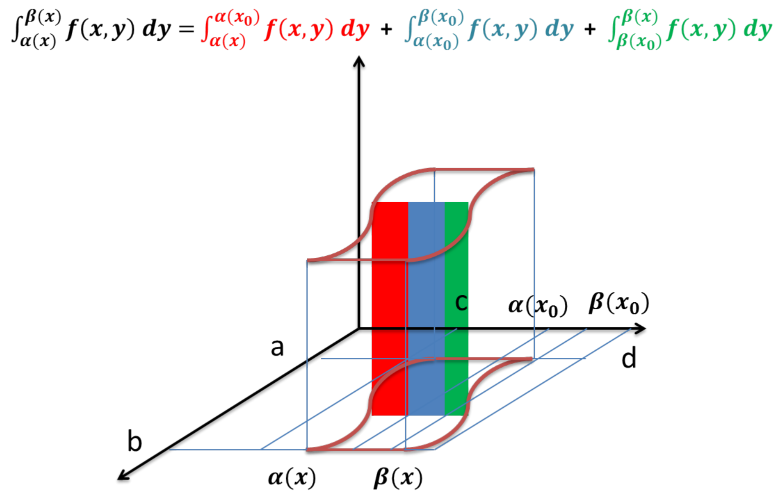
$$F'(x)=\int_{\alpha(x)}^{\beta(x)}\frac{\partial f(x,y)}{\partial x}\ dy+f(x,\beta(x))\beta'(x)-f(x,\alpha(x))\alpha'(x)$$

Demostración. Sea $x_0\in[a,b]$ entonces podemos escribir

$$F(x)=\int_{\alpha(x)}^{\beta(x)} f(x,y)\ dy=\int_{\alpha(x)}^{\alpha(x_0)} f(x,y)\ dy+\int_{\alpha(x_0)}^{\beta(x_0)} f(x,y)\ dy+\int_{\beta(x_0)}^{\beta(x)} f(x,y)\ dy$$

y se tiene que

$$F(x_0)=\int_{\alpha(x_0)}^{\alpha(x_0)} f(x_0,y)\ dy+\int_{\alpha(x_0)}^{\beta(x_0)} f(x_0,y)\ dy+\int_{\beta(x_0)}^{\beta(x_0)} f(x_0,y)\ dy$$



hacemos el cociente

$$\frac{F(x)-F(x_0)}{x-x_0} = \frac{1}{x-x_0}\int_{\alpha(x)}^{\alpha(x_0)} f(x,y)\ dy + \int_{\alpha(x_0)}^{\beta(x_0)} \frac{f(x,y)-f(x_0,y)}{x-x_0}\ dy + \frac{1}{x-x_0}\int_{\beta(x_0)}^{\beta(x)} f(x,y)\ dy$$

tenemos que

$$-\frac{1}{x-x_0}\int_{\alpha(x_0)}^{\alpha(x)} f(x,y)\ dy = -\frac{\alpha(x)-\alpha(x_0)}{x-x_0} f(x,\overline{x}) \quad \overline{x}\in[\alpha(x),\alpha(x_0)] \underbrace{=}_{x\to x_0} -\alpha'(x_0)(f(x_0,\alpha(x_0)))$$

$$\int_{\alpha(x_0)}^{\beta(x_0)} \frac{f(x,y)-f(x_0,y)}{x-x_0}\ dy \underbrace{=}_{x\to x_0} \int_{\alpha(x_0)}^{\beta(x_0)} \frac{\partial f(x,y)}{\partial x} dy$$

$$\frac{1}{x-x_0}\int_{\beta(x_0)}^{\beta(x)} f(x,y)\ dy = \frac{\beta(x)-\beta(x_0)}{x-x_0} f(x,\overline{x}) \quad \overline{x}\in[\beta(x_0),\beta(x)] \underbrace{=}_{x\to x_0} \beta'(x_0)(f(x_0,\beta(x_0)))$$

al sumar lo anterior se obtiene el resultado

Ejemplo Dada la función

$$F(x) = \int_0^x \operatorname{sen}(xy)\ dy$$

calcular $F'(x)$

Solución En este caso hacemos

$$f(x,y) = \operatorname{sen}(xy)$$

$$\alpha(x) = 0, \quad \beta(x) = x$$

y según la fórmula de Leibniz

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\int_0^x \operatorname{sen}(xy)\, dy\right)$$

$$= \int_0^x \left(\frac{\partial}{\partial x}\operatorname{sen}(xy)\right) dy - \underbrace{0 \cdot \operatorname{sen}(x\cdot 0)}_{\alpha'(x)\ f(x,\alpha(x))} + \underbrace{(1)f(x,x)}_{\beta'(x)f(x,\beta(x))}$$

$$= \int_0^x y\cos(xy)\, dy + \operatorname{sen}(x^2)$$

Ejemplo A partir del resultado

$$\int_0^1 \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right)\Big|_0^1 = \frac{\pi}{4a}$$

calcular las siguientes integrales

$$a)\quad \int_0^a \frac{dx}{(x^2+a^2)^2} \qquad b)\quad \int_0^a \frac{dx}{(x^2+a^2)^3}$$

Solución Para el inciso a se tiene derivando la integral según la regla de Leibniz

$$\int_0^1 \frac{dx}{x^2+a^2} = \frac{\pi}{4a}$$

$$\Rightarrow \frac{d}{da}\left(\int_0^a \frac{dx}{x^2+a^2}\right) = \frac{d}{da}\left(\frac{\pi}{4a}\right)$$

$$\Rightarrow \int_0^a \frac{\partial}{\partial a}\left(\frac{1}{x^2+a^2}\right) dx = \frac{d}{da}\left(\frac{\pi}{4a}\right)$$

Tenemos que

$$\frac{\partial}{\partial a}\left(\frac{1}{x^2+a^2}\right) = \frac{-2a}{(x^2+a^2)^2}$$

$$\frac{d}{da}\left(\frac{\pi}{4a}\right) = -\frac{\pi}{4a^2}$$

por otro lado para $\beta(x) = a$ se tiene

$$f(a,x) = \frac{1}{x^2+a^2} \;\Rightarrow\; f(a,\beta(x)) = \frac{1}{a^2+a^2} = \frac{1}{2a^2}$$

por lo tanto

$$f(x,\beta(x))\beta'(x) = \frac{1}{2a^2}(a)' = \frac{1}{2a^2}$$

y para $\alpha(x) = 0$ se tiene

$$f(a,x) = \frac{1}{x^2+a^2} \;\Rightarrow\; f(a,\alpha(x)) = \frac{1}{0^2+a^2}(0)' = \frac{1}{a^2}(0) = 0$$

por lo tanto

$$f(x,\alpha(x))\alpha'(x)=\frac{1}{a^2}(0)'=0$$

tenemos entonces que

$$\int_0^a \frac{-2a}{(x^2+a^2)^2}\,dx+\frac{1}{2a^2}=-\frac{\pi}{4a^2} \;\Rightarrow\; \int_0^a \frac{dx}{(x^2+a^2)^2}=\frac{2+\pi}{8a^3}$$

Para el inciso b se tiene al derivar lo anterior

$$\frac{d}{dx}\left(\int_0^a \frac{dx}{(x^2+a^2)^2}\right)=\frac{d}{dx}\left(\frac{2+\pi}{8a^3}\right) \;\Rightarrow\; \int_0^a \frac{-4a}{(x^2+a^2)^3}\,dx+\frac{1}{4a^2}=-\frac{3(2+\pi)}{8a^4} \;\Rightarrow\; \int_0^a \frac{dx}{(x^2+a^2)^3}=\frac{8+3\pi}{32a^5}$$

□