Derivación bajo el signo de integral

Teorema 1. *Sea* $f : [a,b] \times [c,d] \to \mathbb{R}$ *continua tal que* $\dfrac{\partial f}{\partial y}$ *existe y es continua en* $[a,b] \times [c,d]$. *Definimos* $g : [c,d] \to \mathbb{R}$ *como*

$$g(y) = \int_a^b f(x,y)\ dx$$

entonces g *es derivable en* $[c,d]$ *y además*

$$g'(y) = \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx$$

Demostración. Sabemos que $\forall\ (x,y) \in [a,b] \times [c,d]$

$$\int_c^y \frac{\partial f}{\partial y}(x,t)\ dt = f(x,y) - f(x,c)$$

de donde

$$f(x,y) = \int_c^y \frac{\partial f}{\partial y}(x,t)\ dt + f(x,c)$$

por lo tanto

$$g(y) = \int_a^b f(x,y)\ dx$$

$$= \int_a^b \left(\int_c^y \frac{\partial f}{\partial y}(x,t)\ dt + f(x,c) \right)\ dx$$

$$= \int_a^b \left(\int_c^y \frac{\partial f}{\partial y}(x,t)\ dt \right)\ dx + \int_a^b f(x,c)\ dx$$

aplicamos el teorema de fubini

$$= \int_c^y \left(\int_a^b \frac{\partial f}{\partial y}(x,t)\ dx \right)\ dt + \int_a^b f(x,c)\ dx$$

Derivando con respecto a y se tiene

$$\frac{\partial}{\partial y} \left(\int_c^y \left(\int_a^b \frac{\partial f}{\partial y}(x,t)\ dx \right)\ dt + \int_a^b f(x,c)\ dx \right)$$

$$= \int_a^b \frac{\partial f}{\partial y}(x,y)\ dx$$

□

Ejemplo Calcular la integral

$$\int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + a^2\cos^2(x)\right) dx \quad a \neq 0$$

Solución Para esto definimos la función $f(x,y) = \ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right)$ la cual es derivable y continua y defino

$$g(y) = \int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right) dx$$

Vamos a trabajar sobre esta $g(y)$ usando derivación bajo integral, tenemos entonces que

$$g'(y) = \frac{\partial}{\partial y}\int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right) dx = \int_0^{\frac{\pi}{2}} \frac{\partial}{\partial y}\ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right) dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{2y\cos^2(y)}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx = \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{\cos^2(x)(y^2-1)}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx = \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{\cos^2(x)y^2 - \cos^2(x)}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx$$

$$= \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \frac{\operatorname{sen}^2(x) + \cos^2(x)y^2 - \cos^2(x) - \operatorname{sen}^2(x)}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx = \frac{2y}{y^2-1}\int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{\operatorname{sen}^2(x) + y^2\cos^2(x)}\right) dx$$

$$= \frac{2y}{y^2-1}\left(\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx\right)$$

Esta última integral la trabajamos aparte de la siguiente forma haciendo en cambio de variable $u = \tan(x)$ por lo que se tiene

$$\int_0^{\frac{\pi}{2}} \frac{1}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos^2(x)\left(\frac{\operatorname{sen}^2(x)}{\cos^2(x)} + y^2\right)} dx = \int_0^{\infty} \frac{du}{u^2+y^2} = \frac{1}{y^2}\int_0^{\infty} \frac{du}{\left(\frac{u}{y}\right)^2 + 1}$$

$$= \frac{1}{y}\arctan\left(\frac{u}{y}\right)\Big|_0^{\infty} = \frac{\pi}{2y}$$

Por lo tanto

$$\frac{2y}{y^2-1}\left(\frac{\pi}{2} - \int_0^{\frac{\pi}{2}} \frac{1}{\operatorname{sen}^2(x) + y^2\cos^2(x)} dx\right) = \frac{2y}{y^2-1}\left(\frac{\pi}{2} - \frac{\pi}{2y}\right) = \frac{\pi}{y+1}$$

Tenemos entonces

$$g'(y) = \frac{\pi}{y+1} \Rightarrow g(y) = \pi\ln(y+1) + C$$

Ahora por un lado

$$g(y) = \int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right) dx \Rightarrow g(1) = 0$$

Por otro lado

$$g(1) = \pi\ln(2) + C$$

Por lo tanto

$$\pi\ln(2) + C = 0 \Rightarrow C = -\pi\ln(2)$$

de esta manera

$$g(y) = \pi \ln(y+1) - \pi \ln(2) \Rightarrow g(y) = \pi \ln\left(\frac{y+1}{2}\right)$$

Y regresando a nuestra integral

$$\int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + y^2\cos^2(x)\right) dx = \pi \ln\left(\frac{y+1}{2}\right)$$

Por lo tanto

$$\int_0^{\frac{\pi}{2}} \ln\left(\operatorname{sen}^2(x) + a^2\cos^2(x)\right) dx = \pi \ln\left(\frac{a+1}{2}\right)$$