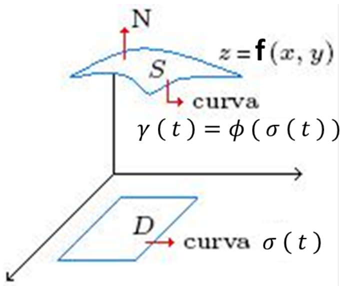
Teorema de Stokes



El teorema de Stokes relaciona la integral de línea de un campo vectorial alrededor de una curva cerrada simple $\gamma \in \mathbb{R}^3$, con la integral sobre una superficie de la cual γ es la frontera. Es decir, si se tiene S una superficie orientada con vector normal unitario N y frontera una curva cerrada γ y un campo vectorial F de clase C^1 se cumple que

$$\int_\gamma F = \int\int_S (rotF \cdot N) dS$$

Si la superficie S es la gráfica de una función $z = f(x,y)$ con (x,y) variando en una región D del plano xy, que tiene derivadas parciales segundas continuas, podemos parametrizarla por medio de la función $\phi : \mathbb{R}^2 \to \mathbb{R}^3$ dada por

$$\phi(u,v) = (u, v, f(u,v))$$

Mientras que el campo $F : \mathbb{R}^3 \to \mathbb{R}^3$ esta dado por

$$F(x,y,z) = P(x,y,z)\widehat{i} + Q(x,y,z)\widehat{j} + R(x,y,z)\widehat{k}$$

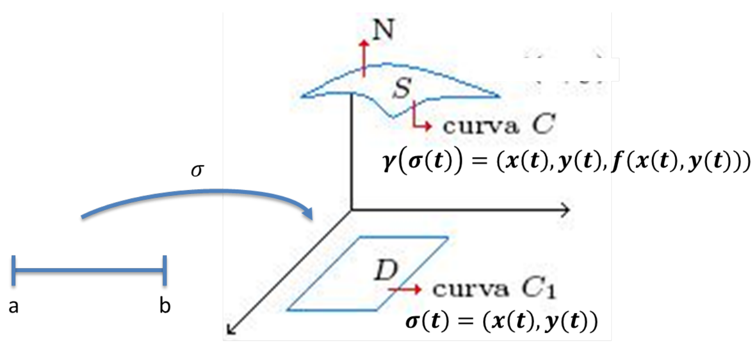
donde P, Q y R tienen parciales primeras continuas
Ahora para la frontera de la superficie S consideramos la curva $\sigma : \mathbb{R} \to \mathbb{R}^2$ dada por

$$\sigma(t) = (x(t), y(t))$$

de esta manera la frontera de la superficicie S sera la función $\gamma : \mathbb{R}^2 \to \mathbb{R}^3$ dada por

$$\gamma(t) = \phi(\sigma(t)) = \phi(x(t), y(t)) = x(t)\widehat{i} + y(t)\widehat{j} + z(t) = f(x(t), y(t))\widehat{k}$$

con $a \leq t \leq b$



tenemos entonces que

$$\int_\gamma F = \int_a^b F(\gamma(t)) \cdot \gamma'(t)dt = \int_a^b \ P(\gamma(t), Q(\gamma(t)), R(\gamma(t))) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}\right) \ dt =$$

$$\int_a^b \left[P(\gamma(t))\frac{dx}{dt} + Q(\gamma(t))\frac{dy}{dt} + R(\gamma(t))\left(\frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}\right)\right] dt$$

$$= \int_a^b \left[\left(P(\gamma(t)) + R(\gamma(t))\frac{\partial z}{\partial x}\right)\frac{dx}{dt} + \left(Q(\gamma(t)) + R(\gamma(t))\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t}\right] dt =$$

$$\int_\gamma \left(P(\gamma(t)) + R(\gamma(t))\frac{\partial z}{\partial x}\right) dx + \left(Q(\gamma(t)) + R(\gamma(t))\frac{\partial z}{\partial y}\right) dy \underbrace{=}_{Green}$$

$$\int\int_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA =$$

Al ser $u = x$ y $v = y$, se tiene que $z = f(x,y) = f(u,v)$ se tiene

$$P = P(x,y,z) = P(u,v,z) \Rightarrow_{Regla\ Cadena} \frac{\partial P}{\partial y} = \frac{\partial P}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial P}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y}$$

$$Q = Q(x,y,z) = Q(u,v,z) \Rightarrow_{Regla\ Cadena} \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Q}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial x}{\partial x}$$

$$R = R(x,y,z) = R(u,v,z) \Rightarrow_{Regla\ Cadena} \frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x} = \frac{\partial R}{\partial x} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial x}$$

$$R = R(x,y,z) = R(u,v,z) \Rightarrow_{Regla\ Cadena} \frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y} = \frac{\partial R}{\partial y} + \frac{\partial R}{\partial z}\frac{\partial z}{\partial y}$$

por lo tanto

$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) = \frac{\partial Q}{\partial x} + R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} = \frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial z}{\partial x} + R\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial x} + \frac{\partial z}{\partial y}\frac{\partial R}{\partial z}\frac{\partial z}{\partial x}$$

$$\frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right) = \frac{\partial P}{\partial y} + R\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\frac{\partial R}{\partial y} = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} + R\frac{\partial^2 z}{\partial y \partial x} + \frac{\partial z}{\partial x}\frac{\partial R}{\partial y} + \frac{\partial z}{\partial x}\frac{\partial R}{\partial z}\frac{\partial z}{\partial y}$$

Por tener $z = f(x,y)$ se tiene $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial^2 z}{\partial y \partial x}$; entonces

$$\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial x}\left(P + R\frac{\partial z}{\partial x}\right) = -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

por lo tanto

$$\int\int_D \left[\frac{\partial}{\partial x}\left(Q + R\frac{\partial z}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial z}{\partial x}\right)\right] dA =$$

$$\int\int_D \left[-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\right] dA$$

Calculamos ahora $\displaystyle\int\int_S rotF \cdot NdS$

$$rotF = \nabla \times F = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\ ,\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\ ,\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

$$N = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ 1 & 0 & \dfrac{\partial z}{\partial x} \\ 0 & 1 & \frac{\partial z}{\partial y} \end{vmatrix} = \left(-\frac{\partial z}{\partial x} , -\frac{\partial z}{\partial y} , 1 \right)$$

$$\therefore \quad rotF \cdot N = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} , \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} , \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \cdot \left(-\frac{\partial z}{\partial x} , -\frac{\partial z}{\partial y} , 1 \right) =$$

$$= -\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\therefore \quad \int\int_D rotF \cdot N dS = \int\int_D \left(-\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \frac{\partial z}{\partial x} - \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \frac{\partial z}{\partial y} + \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\therefore \quad \int\int_S rotF \cdot N dS = \int_\gamma F \cdot dR$$

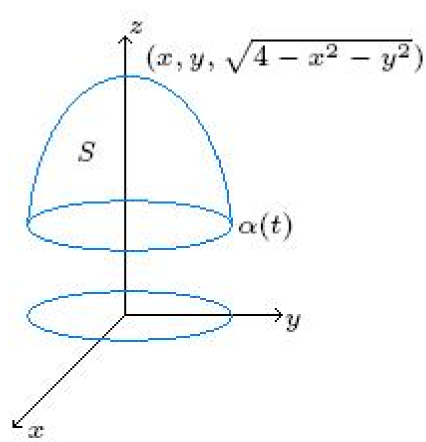
Ejemplo: Comprobar Stokes calculando el flujo del rotacional del campo

$$F(x, y, z) = (y - 2x, yz^2, -y^2 z) \quad S = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 4 \quad , \quad z > 1\}$$

Sol. Parametrizamos el casquete

$$r : D \subset \mathbb{R}^2 \to \mathbb{R}^3, \quad (x, y) = (x, y, \sqrt{4 - x^2 - y^2})$$

de esta forma $S = r(D)$ donde $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < 3\}$



se tiene que el vector normal es

$$N(x, y) = \frac{\partial r}{\partial x} \times \frac{\partial r}{\partial y} = \left(\frac{x}{\sqrt{4 - x^2 - y^2}} , \frac{y}{\sqrt{a - x^2 - y^2}} , 1 \right)$$

Y el rotacional es $rotF = \nabla \times F = \begin{vmatrix} \widehat{i} & \widehat{j} & \widehat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y-2x & yz^2 & -y^2z \end{vmatrix} = (-4yz, 0, -1)$

$\therefore \int\int_S rotF \cdot N dxdy = \int\int_D (-4xy-1)dxdy =$

$= \int_{-\sqrt{3}}^{\sqrt{3}} \int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} (-4xy-1)dxdy \int_{-\sqrt{3}}^{\sqrt{3}} -2x^2y - x \Big|_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}} dy =$

$= \int_{-\sqrt{3}}^{\sqrt{3}} -2(3-y^2)y - \sqrt{3-y^2} - \left[-2(-\sqrt{3-y^2})\right] y - \sqrt{3-y^2} = = \int_{-\sqrt{3}}^{\sqrt{3}} -2\sqrt{3-y^2} = -2\int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{3-y^2}dy =$
$-2\pi\dfrac{(\sqrt{3})^2}{2} = -3\pi$

Para comprobar Stokes calculamos la integral curvilinea.

$\int_\alpha (y-2x)dx + yz^2dy - y^2zdz$ Siendo $\alpha = r \circ \gamma$, $\gamma(t) = (\sqrt{3}cos(t), \sqrt{3}sen(t))$

$\therefore \alpha : [0, 2\pi] \to \mathbb{R}^3$ con $\alpha(t) = r(\gamma(t)) = (\sqrt{3}cos(t) , \sqrt{3}sen(t) , 1)$

$\therefore$ la integral vale

$\int_\alpha (y-2x)dx + yz^2dy - y^2zdz = \int_0^{2\pi} F(\alpha(t)) \cdot \alpha'(t)dt =$

$= \int_0^{2\pi} (\sqrt{3}sen(t) - 2\sqrt{3}cos(t) , \sqrt{3}sen(t) , -3sen^2(t)) \cdot (-\sqrt{3}sen(t) , \sqrt{3}cos(t) , 0) =$

$= \int_0^{2\pi} (-3sen^2(t) + 9sen(t)cos(t)dt = -3\pi$