

Diferenciales de funciones $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$

Tenemos que $f : A \subset \mathbb{R}^2 \rightarrow \mathbb{R}$ es diferenciable si

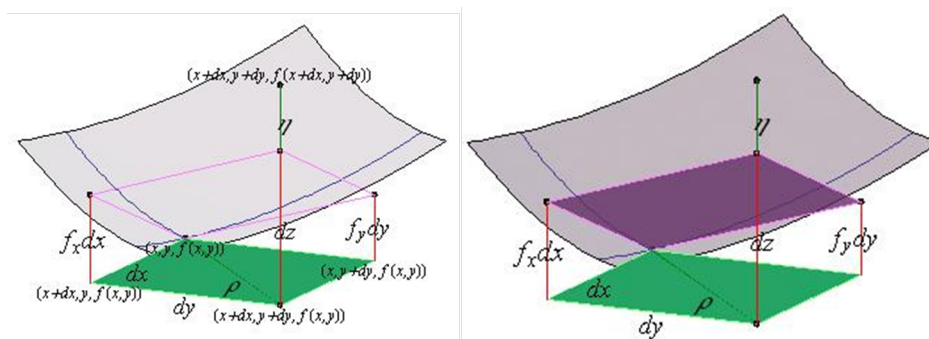
$$f(x_0 + h_1, y_0 + h_2) = f(x_0, y_0) + \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2 + r(h_1, h_2)$$

cumple

$$\lim_{(h_1, h_2) \rightarrow (0,0)} \frac{r(h_1, h_2)}{\|(h_1, h_2)\|} = 0$$

Esto se puede escribir como

$$f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)h_1 + \frac{\partial f}{\partial y}(x_0, y_0)h_2 + r(h_1, h_2)$$



tomando

$$\begin{aligned} f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) &= \Delta z \\ \frac{\partial f}{\partial x}(x_0, y_0)h_1 &= \frac{\partial f}{\partial x}(x_0, y_0)\Delta x \\ \frac{\partial f}{\partial y}(x_0, y_0)h_2 &= \frac{\partial f}{\partial y}(x_0, y_0)\Delta y \end{aligned}$$

tenemos que

$$\Delta z = \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y + r(\Delta x, \Delta y)$$

haciendo $\Delta x, \Delta y \rightarrow 0$ tenemos

$$dz = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy$$

Definición 1. Si $z = f(x, y)$ es una función diferenciable, la diferencial de f denotada dz se define

$$dz = \frac{\partial f}{\partial x}(x_0, y_0)dx + \frac{\partial f}{\partial y}(x_0, y_0)dy$$

Ejemplo Calcular la diferencial de $z = 4x^2 - xy$

En este caso

$$dz = \frac{\partial(4x^2 - xy)}{\partial x} dx + \frac{\partial(4x^2 - xy)}{\partial y} dy = (8x - y)dx - xdy$$

Ahora bien

$$f(x_0 + h_1, y_0 + h_2) - f(x_0, y_0) = \Delta z \approx \frac{\partial f}{\partial x}(x_0, y_0)\Delta x + \frac{\partial f}{\partial y}(x_0, y_0)\Delta y$$

expresa el cambio aproximado de $z = f(x, y)$ cuando (x, y) pasa a $(x + \Delta x, y + \Delta y)$

Ejemplo Aproximar el cambio de $z = 4x^2 - xy$ cuando (x, y) pasa de $(2, 1)$ a $(2, 1, 1, 5)$

En este caso tomamos $x_0 = 2, y_0 = 1, \Delta x = 0,1$ y $\Delta y = ,5$ y el valor de cambio será

$$\frac{\partial f}{\partial x}(2, 1)\Delta x + \frac{\partial f}{\partial y}(2, 1)\Delta y = (15)(0,1) - 2(0,5) = 1,5$$

mientras que

$$f(2, 1, 1, 5) - f(2, 1) = 14,49 - 14 = 0,49$$

por lo tanto en la aproximacion se cometiio un error de 0,01

Ejemplo Usando diferenciales se quiere calcular aproximadamente

$$A = \frac{0,97}{\sqrt{15,05} + \sqrt[3]{0,98}}$$

Solución Considerando la función

$$f(x, y, z) = \frac{x}{\sqrt{y + \sqrt[3]{z}}}$$

con $x = 1, y = 15, z = 1, dx = -0,03, dy = 0,05$ y $dz = -0,02$ se tiene

$$f(x + dx, y + dy, z + dz) = f(x, y, z) + df(x, y, z)$$

en este caso

$$f(x, y, z) = f(1, 15, 1) = \frac{1}{4}$$

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{y + \sqrt[3]{z}}}, \quad \frac{\partial f}{\partial y} = -\frac{x}{2} (y + \sqrt[3]{z})^{-\frac{3}{2}}, \quad \frac{\partial f}{\partial z} = -\frac{x}{2} (y + \sqrt[3]{z})^{-\frac{3}{2}} \frac{1}{3} z^{-\frac{2}{3}}$$

evaluando en $(1, 15, 1)$ se tiene

$$\frac{\partial f}{\partial x}(1, 15, 1) = \frac{1}{4}, \quad \frac{\partial f}{\partial y}(1, 15, 1) = -\frac{1}{128}, \quad \frac{\partial f}{\partial z}(1, 15, 1) = -\frac{1}{384}$$

de modo que

$$df(1, 15, 1) = \frac{1}{4}(-0,03) - \frac{1}{128}(0,05) - \frac{1}{384}(-0,02) = -\frac{3,01}{384}$$

por lo que

$$A = \frac{1}{4} - \frac{3,01}{384} = 0,242161$$

(el valor es 0,2421726)



Diferencial de orden 2

Si $df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$ entonces una diferencial de orden 2 sería:

$$\begin{aligned} d^2 f &= d(df) = d\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) dx + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy\right) dy \\ &= \left(\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial x \partial y} dy\right) dx + \left(\frac{\partial^2 f}{\partial y \partial x} dx + \frac{\partial^2 f}{\partial y^2} dy\right) dy = \frac{\partial^2 f}{\partial x^2} dx^2 + \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y \partial x} dy dx + \frac{\partial^2 f}{\partial y^2} dy^2 \\ &= \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2 \end{aligned}$$

Por lo tanto

$$d^2 f = d(df) = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$$

Ejemplo Hallar la diferencial de orden 2 para $f(x, y) = e^{x^2+y^2}$

Solución En este caso tenemos la fórmula

$$d^2 f = d(df) = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$$

vamos a calcular las derivadas parciales correspondientes

$$\begin{aligned} \frac{\partial(e^{x^2+y^2})}{\partial x} &= 2xe^{x^2+y^2} \\ \frac{\partial(e^{x^2+y^2})}{\partial y} &= 2ye^{x^2+y^2} \\ \frac{\partial^2(e^{x^2+y^2})}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial(e^{x^2+y^2})}{\partial x}\right) = \frac{\partial(2xe^{x^2+y^2})}{\partial x} = 4x^2e^{x^2+y^2} + 2e^{x^2+y^2} \\ \frac{\partial^2(e^{x^2+y^2})}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial(e^{x^2+y^2})}{\partial y}\right) = \frac{\partial(2ye^{x^2+y^2})}{\partial y} = 4y^2e^{x^2+y^2} + 2e^{x^2+y^2} \\ \frac{\partial^2(e^{x^2+y^2})}{\partial y \partial x} &= \frac{\partial}{\partial y} \left(\frac{\partial(e^{x^2+y^2})}{\partial x}\right) = \frac{\partial(2xe^{x^2+y^2})}{\partial y} = 4xye^{x^2+y^2} \\ \frac{\partial^2(e^{x^2+y^2})}{\partial x \partial y} &= \frac{\partial}{\partial x} \left(\frac{\partial(e^{x^2+y^2})}{\partial y}\right) = \frac{\partial(2ye^{x^2+y^2})}{\partial x} = 4xye^{x^2+y^2} \end{aligned}$$

y la diferencial de orden 2 sería:

$$d^2 f = \left(4x^2e^{x^2+y^2} + 2e^{x^2+y^2}\right) dx^2 + 8xye^{x^2+y^2} dx dy + \left(4y^2e^{x^2+y^2} + 2e^{x^2+y^2}\right) dy^2$$



Diferencial de orden 3

Si $d^2 f = \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2$ entonces una diferencial de orden 3 seria:

$$\begin{aligned} d^3 f &= d(d^2 f) = d \left(\frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2 \right) = \\ &= \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2 \right) dx + \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial y^2} dy^2 \right) dy = \\ &= \left(\frac{\partial^3 f}{\partial x^3} dx^2 + 2 \frac{\partial^3 f}{\partial x^2 \partial y} dx dy + \frac{\partial^3 f}{\partial x \partial y^2} dy^2 \right) dx + \left(\frac{\partial^3 f}{\partial x^2 \partial y} dx^2 + 2 \frac{\partial^3 f}{\partial x \partial y^2} dx dy + \frac{\partial^3 f}{\partial y^3} dy^2 \right) dy = \\ &= \frac{\partial^3 f}{\partial x^3} dx^3 + 2 \frac{\partial^3 f}{\partial x^2 \partial y} dx^2 dy + \frac{\partial^3 f}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f}{\partial x^2 \partial y} dy dx^2 + 2 \frac{\partial^3 f}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f}{\partial y^3} dy^3 = \\ &= \frac{\partial^3 f}{\partial x^3} dx^3 + 3 \frac{\partial^3 f}{\partial x^2 \partial y} dx^2 dy + 3 \frac{\partial^3 f}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f}{\partial y^3} dy^3 \end{aligned}$$

Por lo tanto

$$d^3 f = d(d^2 f) = \frac{\partial^3 f}{\partial x^3} dx^3 + 3 \frac{\partial^3 f}{\partial x^2 \partial y} dx^2 dy + 3 \frac{\partial^3 f}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f}{\partial y^3} dy^3$$

Diferencial de orden 4

Si $d^3 f = \frac{\partial^3 f}{\partial x^3} dx^3 + 3 \frac{\partial^3 f}{\partial x^2 \partial y} dx^2 dy + 3 \frac{\partial^3 f}{\partial x \partial y^2} dx dy^2 + \frac{\partial^3 f}{\partial y^3} dy^3$ entonces una diferencial de orden 4 seria:

$$d^4 f = d(d^3 f) = \frac{\partial^4 f}{\partial x^4} dx^4 + 4 \frac{\partial^4 f}{\partial x^3 \partial y} dx^3 dy + 6 \frac{\partial^4 f}{\partial x^2 \partial y^2} dx^2 dy^2 + 4 \frac{\partial^4 f}{\partial x \partial y^3} dx dy^3 + \frac{\partial^4 f}{\partial y^4} dy^4$$

Diferencial de orden n

$$d^n f = \frac{\partial^n f}{\partial x^n} dx^n + \binom{n}{1} \frac{\partial^{n-1} f}{\partial x^{n-1} \partial y} dx^{n-1} dy + \binom{n}{2} \frac{\partial^{n-2} f}{\partial x^{n-2} \partial y^2} dx^{n-2} dy^2 + \dots + \binom{n}{k} \frac{\partial^{n-k} f}{\partial x^{n-k} \partial y^k} dx^{n-k} dy^k + \dots + \frac{\partial^n f}{\partial y^n} dy^n$$

que se puede escribir

$$d^n f = \sum_{j=0}^n \binom{n}{j} \frac{\partial^n f}{\partial x^{n-j} \partial y^j} dx^{n-j} dy^j$$

