Diferencial de orden n

$$d^n f = \frac{\partial^n f}{\partial x^n}dx^n + \binom{n}{1}\frac{\partial^{n-1} f}{\partial x^{n-1}\partial y}dx^{n-1}dy + \binom{n}{2}\frac{\partial^{n-2} f}{\partial x^{n-2}\partial y^2}dx^{n-2}dy^2 + \cdots + \binom{n}{k}\frac{\partial^{n-k} f}{\partial x^{n-k}\partial y^k}dx^{n-k}dy^k + \cdots + \frac{\partial^n f}{\partial y^n}dy^n$$

que se puede escribir

$$d^n f = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j$$

Ejercicio Probar usando inducción

$$d^n f = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j$$

Solución *Demostración.* Para n=1 se tiene

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

Suponemos valido para n

$$d^n f = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j$$

Por demostrar que es valida para n+1

$$d^{n+1} f = d(d^n f) = \frac{\partial}{\partial x}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j\right)dx + \frac{\partial}{\partial y}\left(\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j\right)dy =$$

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n-j}\partial y^{j+1}}dx^{n-j}dy^{j+1} =$$

$$\sum_{j=0}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=1}^{n+1}\binom{n}{j-1}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\binom{n}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \sum_{j=1}^{n}\binom{n}{j-1}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\left(\binom{n}{j} + \binom{n}{j-1}\right)\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} =$$

$$\frac{\partial^{n+1} f}{\partial x^{n+1}}dx^{n+1} + \sum_{j=1}^{n}\binom{n+1}{j}\frac{\partial^{n+1} f}{\partial x^{n+1-j}\partial y^j}dx^{n+1-j}dy^j + \frac{\partial^{n+1} f}{\partial y^{n+1}}dy^{n+1} = \sum_{j=0}^{n+1}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}dx^{n-j}dy^j$$

□

La última fórmula puede expresarse simbólicamente por la ecuación

$$d^n f = \left(\frac{\partial}{\partial x} \ dx + \frac{\partial}{\partial y} \ dy \right)^n f$$

donde primero debe desarrollarse le expresión de la derecha formalmente por medio del teorema del binomio y, a continuación deben sustituirse los términos

$$\frac{\partial^n f}{\partial x^n} dx^n, \frac{\partial^n f}{\partial x^{n-1} \partial y} dx^{n-1} dy, \cdots, \frac{\partial^n f}{\partial y^n} dy^n$$

por los términos

$$\left(\frac{\partial}{\partial x} dx \right)^n f, \left(\frac{\partial}{\partial x} dx \right)^{n-1} \left(\frac{\partial}{\partial y} dy \right) f, \cdots, \left(\frac{\partial}{\partial y} dy \right)^n f$$

Teorema de Taylor para funciones $f : A \subset \mathbb{R}^2 \to \mathbb{R}$

Recordando el Teorema de Taylor para funciones $f : \mathbb{R} \to \mathbb{R}$

Teorema 1. *Si $f(x)$ tiene n-ésima derivada continua en una vecindad de x_0, entonces en esa vecindad*

$$f(x) = f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \frac{1}{3!} f'''(x_0)(x - x_0)^3 + \ldots + \frac{1}{n!} f^n(x_0)(x - x_0)^n + R_n$$

donde

$$R_n = \frac{f^{n+1}(\epsilon)}{(n+1)!} (x - x_0)^{n+1}, \quad \textit{donde} \quad \epsilon \in (x_0, x)$$

Sea $f : A \subset \mathbb{R}^2 \to \mathbb{R}$ y sea $F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$ con $t \in [0, 1]$, de esta manera f recorre el segmento de $[x_0, y_0]$ a $[x_0 + h_1 t, y_0 + h_2 t]$. Se tiene entonces que usando la regla de la cadena

$$F'(t) = \frac{\partial f}{\partial x}(x_0 + h_1 t, y_0 + h_2 t) \cdot \frac{d(x_0 + h_1 t)}{dt} + \frac{\partial f}{\partial y}(x_0 + h_1 t, y_0 + h_2 t) \cdot \frac{d(y_0 + h_2 t)}{dt} =$$

$$\frac{\partial f}{\partial x}(x_0 + h_1 t, y_0 + h_2 t) \cdot h_1 + \frac{\partial f}{\partial y}(x_0 + h_1 t, y_0 + h_2 t) \cdot h_2$$

Vamos ahora a calcular $F''(t)$

$$F''(t) = \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 \right) h_1 + \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} h_1 + \frac{\partial f}{\partial y} h_2 \right) h_2 =$$

$$\frac{\partial^2 f}{\partial x^2} h_1^2 + 2 \frac{\partial^2 f}{\partial x \partial y} h_1 h_2 + \frac{\partial^2 f}{\partial y^2} h_2^2$$

simbólicamente se puede escribir

$$F''(t) = \left(\frac{\partial}{\partial x} \cdot h_1 + \frac{\partial}{\partial y} \cdot h_2 \right)^2 f$$

y en general

$$F^n(t) = \frac{\partial^n f}{\partial x^n}h_1^n + \binom{n}{1}\frac{\partial^{n-1} f}{\partial x^{n-1}\partial y}h_1^{n-1}h_2 + \binom{n}{2}\frac{\partial^{n-2} f}{\partial x^{n-2}\partial y^2}h_1^{n-2}h_2^2 + \cdots + \binom{n}{k}\frac{\partial^{n-k} f}{\partial x^{n-k}\partial y^k}h_1^{n-k}h_2^k + \cdots + \frac{\partial^n f}{\partial y^n}h_2^n$$

que simbólicamente se puede escribir

$$F^n = \sum_{j=0}^{n}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}h_1^{n-j}h_2^j = \left(\frac{\partial}{\partial x}\cdot h_1 + \frac{\partial}{\partial y}\cdot h_2\right)^n f$$

Ahora bien si se aplica la fórmula de Taylor con la forma del residuo de Lagrange a la función

$$F(t) = f(x_0 + h_1 t, y_0 + h_2 t)$$

y ponemos $t = 0$, se tiene

$$F(t) = F(0) + \frac{1}{1!}F'(0)t + \frac{1}{2!}F''(0)t^2 + \frac{1}{3!}F'''(0)t^3 + ... + + \frac{1}{n!}F^{^n}(0)t^n + R_n$$

ahora bien con $t = 1$

$$f(x_0+h_1, y_0+h_2) = f(x_0, y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0, y_0)\cdot h_1 + \frac{\partial f}{\partial y}(x_0, y_0)\cdot h_2\right) + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0, y_0)h_1^2 + 2\frac{\partial^2 f}{\partial x\partial y}(x_0, y_0)h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)\right.$$

$$+ \cdots + \frac{1}{n!}\left(\sum_{j=0}^{n+1}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}(x_0, y_0)h_1^{n-j}h_2^j\right)$$

$x = x_0 + h_1$, $y_0 + h_2 = y$ por lo que $h_1 = x - x_0$ y $h_2 = y - y_0$ entonces

$$f(x, y) = f(x_0, y_0) + \frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0, y_0)\cdot(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0)\cdot(y - y_0)\right) +$$

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0, y_0)(x - x_0)^2 + 2\frac{\partial^2 f}{\partial x\partial y}(x_0, y_0)(x - x_0)(y - y_0) + \frac{\partial^2 f}{\partial y^2}(x_0, y_0)(y - y_0)^2\right) +$$

$$\cdots + \frac{1}{n!}\left(\sum_{j=0}^{n+1}\binom{n}{j}\frac{\partial^n f}{\partial x^{n-j}\partial y^j}(x_0, y_0)(x - x_0)^{n-j}(y - y_0)^j\right) + R_n$$

donde

$$R_n = \frac{1}{n + 1!}\left((x - x_0)^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}}(\xi, \eta) + \cdots + (y - y_0)^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}(\xi, \eta)\right)$$

donde $\xi \in (x_0, x_0 + h_1)$ y $\eta \in (y_0, y_0 + h_2)$
En general el residuo R_n se anula en un orden mayor que el término $d^n f$

Ejercicio Desarrollar la fórmula de Taylor en $(x_0, y_0) = (0, 0)$ con $n = 3$ para la función

$$f(x, y) = e^y \cos x$$

Solución En este caso tenemos que

$$f(0,0) = e^0 \cos(0) = 1$$

Para la diferencial de orden 1

$$\frac{\partial f}{\partial x}(0,0) \Rightarrow \frac{\partial (e^y \cos x)}{\partial x}(0,0) \Rightarrow -e^y \operatorname{sen}(x)\big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}(0,0) \Rightarrow \frac{\partial (e^y \cos x)}{\partial y}(0,0) \Rightarrow -e^y \cos(x)\big|_{(0,0)} = 1$$

por lo tanto

$$\frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\right) = \frac{1}{1!}\left((0)(x) + (1)(y)\right) = y$$

Para la diferencial de orden 2

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial x^2}(0,0) \Rightarrow -e^y \cos\ x\big|_{(0,0)} = -1$$

$$\frac{\partial^2 f}{\partial y^2}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial y^2}(0,0) \Rightarrow e^y \cos\ x\big|_{(0,0)} = 1$$

$$\frac{\partial^2 f}{\partial x\ \partial y}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial x\ \partial y}(0,0) \Rightarrow -e^y \operatorname{sen}\ x\big|_{(0,0)} = 0$$

Por lo tanto

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(x_0,y_0)h_1h_2 + \frac{\partial^2 f}{\partial y^2}(x_0,y_0)h_2^2\right) = \frac{1}{2!}((-1)x^2 + 2(0)xy + (1)y^2)$$

Para la diferencial de orden 3

$$\frac{\partial^3 f}{\partial x^3}(x_0,y_0) \Rightarrow \quad \Rightarrow e^y \operatorname{sen}\ x\big|_{(0,0)} = 0$$

$$\frac{\partial^3 f}{\partial y^3}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial y^3}(0,0) \Rightarrow e^y \cos\ x\big|_{(0,0)} = 1$$

$$\frac{\partial^3 f}{\partial x^2\ \partial y}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial x^2\ \partial y}(0,0) \Rightarrow -e^y \cos\ x\big|_{(0,0)} = -1$$

$$\frac{\partial^3 f}{\partial y^3}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial y^3}(0,0) \Rightarrow e^y \cos\ x\big|_{(0,0)} = 1$$

$$\frac{\partial^3 f}{\partial x\ \partial y^2}(x_0,y_0) \Rightarrow \frac{\partial^2 (e^y \cos x)}{\partial x\ \partial y^2}(0,0) \Rightarrow -e^y \operatorname{sen}\ x\big|_{(0,0)} = 0$$

Por lo tanto

$$\frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}h_1^3 + 3\frac{\partial^3 f}{\partial x^2 \partial y}h1^2h_2 + 3\frac{\partial^3 f}{\partial x \partial y^2}h_1h_2^2 + \frac{\partial^3 f}{\partial y^3}h2^3\right) =$$

$$\frac{1}{3!}\left((0)(x^3) + 3(-1)x^2y + 3(0)xy^2 + (1)y^3\right)$$

Finalmente para el residuo se tiene

$$\frac{\partial^4 f}{\partial x^4}(x_0,y_0) \;\Rightarrow\; \frac{\partial^4(e^y\cos(x))}{\partial y^3}(0,0) \;\Rightarrow\; e^y\cos\ x\big|_{(\xi,\eta)} = e^\eta\cos\ \xi$$

$$\frac{\partial^4 f}{\partial x^3\partial y}(x_0,y_0) \;\Rightarrow\; \frac{\partial^4(e^y\cos x)}{\partial x^3\partial y}(0,0) \;\Rightarrow\; e^y\,\text{sen}\ x\big|_{(\xi,\eta)} = e^\eta\,\text{sen}\ xi$$

$$\frac{\partial^4 f}{\partial x^2\partial y^2}(x_0,y_0) \;\Rightarrow\; \frac{\partial^4(e^y\cos x)}{\partial x^2\partial y^2}(0,0) \;\Rightarrow\; -e^y\cos\ x\big|_{(\xi,\eta)} = -e^\eta\cos\ \xi$$

$$\frac{\partial^4 f}{\partial x\partial y^3}(x_0,y_0) \;\Rightarrow\; \frac{\partial^4(e^y\cos x)}{\partial x\partial y^3}(0,0) \;\Rightarrow\; -e^y\,\text{sen}\ x\big|_{(\xi,\eta)} = -e^\eta\,\text{sen}\ \xi$$

$$\frac{\partial^4 f}{\partial y^4}(x_0,y_0) \;\Rightarrow\; \frac{\partial^4(e^y\cos x)}{\partial y^4}(0,0) \;\Rightarrow\; e^y\cos\ x\big|_{(\xi,\eta)} = e^\eta\cos\ \xi$$

$$R_3 = \frac{1}{4!}\left(\frac{\partial^4 f}{\partial x^4}h_1^4 + 4\frac{\partial^4 f}{\partial x^3\partial y}h_1^3h_2 + 6\frac{\partial^4 f}{\partial x^2\partial y^2}h_1^2h_2^2 + 4\frac{\partial^4 f}{\partial x\partial y^3}h_1h_2^3 + \frac{\partial^4 f}{\partial h_2^4}dy^4\right)$$

$$= \frac{1}{4!}\left(x^4e^\eta\cos\ \xi + 4x^3ye^\eta\,\text{sen}\ xi - 6x^2y^2e^\eta\cos\ \xi - 4xy^3e^\eta\,\text{sen}\ \xi + y^4e^\eta\cos\ \xi\right)$$

Por lo que nuestro desarrollo de Taylor nos queda

$$e^y\cos\ x = 1 + y + \frac{1}{2}(x^2-y^2) + \frac{1}{6}(x^3-3xy^2) + R_3$$

donde

$$R_3 = \frac{1}{4!}\left(x^4e^\eta\cos\ \xi + 4x^3ye^\eta\,\text{sen}\ xi - 6x^2y^2e^\eta\cos\ \xi - 4xy^3e^\eta\,\text{sen}\ \xi + y^4e^\eta\cos\ \xi\right)$$

Ejercicio Use la fórmula de Taylor en

$$f(x,y) = \cos\ (x+y)$$

en el punto $(x_0,y_0)=(0,0)$ con $n=2$ para comprobar que

$$\lim_{(x,y)\to(0,0)} \frac{1-\cos\ (x+y)}{(x^2+y^2)^2} = \frac{1}{2}$$

Solución En este caso para

$$f(x,y) = \cos(x+y)$$

se tiene

$$f(0,0) = \cos(0+0) = 1$$

Para la diferencial de orden 1

$$\frac{\partial f}{\partial x}(0,0) \;\Rightarrow\; \frac{\partial(\cos x + y)}{\partial x}(0,0) \;\Rightarrow\; -\text{sen}(x+y)\big|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y}(0,0) \;\Rightarrow\; \frac{\partial(\cos x + y)}{\partial y}(0,0) \;\Rightarrow\; -\text{sen}(x+y)\big|_{(0,0)} = 0$$

por lo tanto

$$\frac{1}{1!}\left(\frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0)\right)=\frac{1}{1!}\left((0)(x)+(0)(y)\right)=0$$

Para la diferencial de orden 2

$$\frac{\partial^2 f}{\partial x^2}(x_0,y_0) \;\Rightarrow\; \frac{\partial^2(\cos x+y)}{\partial x^2}(0,0) \;\Rightarrow\; -\cos\ x+y\big|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial y^2}(x_0,y_0) \;\Rightarrow\; \frac{\partial^2(\cos x+y)}{\partial y^2}(0,0) \;\Rightarrow\; -\cos\ x+y\big|_{(0,0)}=-1$$

$$\frac{\partial^2 f}{\partial x\ \partial y}(x_0,y_0) \;\Rightarrow\; \frac{\partial^2(\cos x+y)}{\partial x\ \partial y}(0,0) \;\Rightarrow\; -\cos\ x+y\big|_{(0,0)}=-1$$

Por lo tanto

$$\frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}(x_0,y_0)h_1^2+2\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)h_1h_2+\frac{\partial^2 f}{\partial y^2}(x_0,y_0)h_2^2\right)=\frac{1}{2!}((-1)x^2-2xy+(-1)y^2)$$

Por lo que nuestro desarrollo de Taylor nos queda

$$\cos(x+y)=1-\frac{x^2}{2}-xy-\frac{y^2}{2}$$

DE manera que

$$\lim_{(x,y)\to(0,0)}\frac{1-\cos\ (x+y)}{(x^2+y^2)^2}=\lim_{(x,y)\to(0,0)}\frac{1-(1-\frac{x^2}{2}-xy-\frac{y^2}{2})}{(x^2+y^2)^2}$$

$$=\lim_{(x,y)\to(0,0)}\frac{1}{2}\frac{(x^2+y^2)^2}{(x^2+y^2)^2}=\frac{1}{2}$$